Sagittal
A Microtonal Notation System

by George D. Secor and David C. Keenan

Introduction

George Secor began development of the Sagittal (pronounced “SAJ-i-tl”) notation system in August 2001. In January 2002 he presented what he had developed to the Yahoo group *tuning* and offered to consider suggestions for improvements. At that stage the system could notate the equal temperaments with 17, 19, 22, 29, 31, 41, and 72 divisions per octave. Little did he know that he had provided a unifying symbolic principle which would ultimately be developed into a system capable of notating almost any conceivable microtonal tuning.

In the early stages of the discussion, one idea that surfaced repeatedly was that a set of symbols indicating prime-number commatic alterations to tones in a Pythagorean series might be used to notate *both* rational intervals (e.g. just intonation) *and* equal divisions of the octave. Similar ideas had previously been proposed by Hermann Helmholtz, Alexander Ellis, Carl Eitz, Paul Rapoport, Daniel Wolf, Joe Monzo, and others[1], but these had never been implemented in a performance notation to such an extent as was being discussed. As a first step, Gene Ward Smith suggested that it would be desirable that there be no more than one comma symbol per prime number, and a selection of 19-limit commas was tentatively identified to define the semantics of the notation.

David Keenan enthusiastically observed that these commas came in an assortment of sizes that made the objective theoretically feasible, but he also realized that devising a set of distinct and recognizable symbols suitable for a performance notation was not going to be a simple matter. For the next year and a half the authors worked together to expand and refine these ideas into a notation system that would be both versatile and powerful, but for which the required complexity would not make it more difficult to do the simpler things. The first year of this process was carried out in the Yahoo groups *tuning* and *tuning-math* and benefited from the input of many other people. The process is recorded in the archives of these groups in all its excruciating detail, complete with numerous dead-ends. Eventually the input from others ceased and we decided to finalize the more esoteric details off-list. The following is an introduction to the resulting system.

[1] See the “HEWM” article in Joe Monzo's *Tonalsoft Encyclopaedia of Tuning* at http://www.tonalsoft.com/enc.

A Triple Feature

The Sagittal notation uses a conventional staff on which the natural notes are in a single series of fifths, with sharps and flats (and doubles thereof) indicating tones that are members of that same series, regardless of the particular tonal system being notated[2]. Therefore, if the notation is used for just intonation, these notes will indicate a Pythagorean tuning. For an equal division of the octave, they will indicate the tones in a series built on that division's best approximation of a fifth.

To indicate alterations to tones in a chain of fifths, the Sagittal notation makes use of new symbols that combine three excellent features of prior notations:

1) Arrows pointing up or down that have been used to indicate alterations in pitch in each direction, most often (but not always) for quartertones;

2) Multiple vertical strokes used by Tartini to indicate multiples of a semi-sharp;

3) Sloping lines used by Bosanquet to indicate commatic alterations in pitch.

Mirrored Arrows

All of the new symbols of the Sagittal notation are various kinds of arrows pointing either up or down to indicate the direction of pitch alteration (the term *Sagittal* being derived from the Latin *sagitta*, arrow[3]). Pairs of symbols that mirror each other vertically indicate equal-but-opposite amounts of pitch alteration, and the apparent size of each symbol generally corresponds to the amount of alteration.

A simple three-segment arrow is used to indicate an alteration in pitch of a *unidecimal diesis* (32:33, ~53.3 cents, approximately a quartertone) in just intonation or its equivalent number of degrees in a temperament. For simplicity we will call this the *11 diesis*. In temperaments in which a sharp or flat alters by an even number of degrees, the 11 diesis will usually be half of this, in which case the symbol is equivalent to a semisharp or semiflat. Such is the case with 72-ET, where the apotome[4] (or sharp/flat) is 6º and the 11 diesis (or semisharp/semiflat) is 3º.

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[2] The Sagittal accidentals may also be used in a consistent manner with systems that do not use a conventional staff, or have more or less than seven nominals, which may not be in a series of fifths, but that is beyond the scope of this article.

[3] In August of 2001, while George Secor was looking through back issues of *Xenharmonikôn* seeking ideas to use in the new notation, he found a comment by Ivor Darreg (*Xenharmonikôn 7&8*, Xenharmonic Bulletin No. 9, Oct. 1978, "The Calmer Mood. 31 Tones/Octave", p. 15), which referred to a sharp symbol with an arrow affixed to the upper left as "sagittarian notation". After puzzling over the meaning for a moment, he concluded that the term would have been more suitable for an archer than for arrows and that the word "sagittal" might be better. Once he had devised the arrow-like symbols for 72-ET with left and right barbs and multiple arrow shafts, he realized that the new name was a perfect fit.

[4] Strictly speaking, an *apotome* is a Pythagorean chromatic semitone, but it is used here as a generic term designating the chromatic semitone occurring in any temperament or division of the octave.

Tartini's Influence

In the first row of symbols in Figure 1, we show the popular combination of Giuseppe Tartini's semi-sharp symbols with Mildred Couper's semi-flat symbols. If quartertones were all that were needed for microtonality, then these are the symbols we would have recommended. Note that we have made the backwards-flat narrower than the forwards-flat to reduce left/right confusability and to make it look more like a *half* flat. Various attempts to extend these symbols in some logical fashion to even finer divisions are, in our opinion, too cumbersome. But while we have rejected these in favor of the other popular notation for quartertones (the up and down arrows), we retain the essential idea from Tartini.

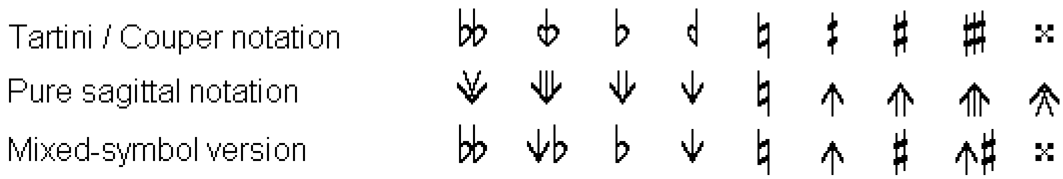

Figure 1 - Comparison of Sagittal Notation with Tartini/Couper Notation

In the Sagittal system Tartini's multiple vertical strokes are combined with the quartertone arrow to produce multi-shaft Sagittal sharp and sesquisharp arrows that are as intuitive as Tartini's symbols, besides being invertible to indicate fractional flats. These are shown in the second row of Figure 1. Instead of four shafts, the double-sharp and double-flat symbols have two shafts that cross to form an elongated "X", one end of which is truncated at the arrowhead. This has the advantage of making them easier to distinguish from the three-shaft sesqui- symbols, besides retaining a resemblance to a conventional double-sharp symbol. To make the triple-shaft easier to distinguish from the double-shaft we vary the spacing between shafts.

If the abandonment of the conventional sharp and flat symbols seems a bit shocking, we should realize that, though they have served us well since they were devised in the Middle Ages, 21st-century microtonality might be better served by something else, and perhaps it is time for an upgrade. We can continue to call these new symbols sharps and flats with semi-, sesqui-, and double- prefixes added as appropriate, inasmuch as it is only the symbols that are changing, not their names or meanings.

However, we recognize that not everyone will agree with this approach, so we have also provided the option of using the Sagittal notation system without abandoning the conventional sharp and flat symbols by presenting the notation in two versions:

1) The mixed-symbol version retains existing sharp and flat symbols and uses only the new single-shaft Sagittal symbols in combination with these, thus eliminating the "Tartini" feature (see the third row of Figure 1). While this version requires fewer total symbols for a music font, it results in a greater number of symbols on a manuscript, which tends to give it a more cluttered appearance when chords are notated. However, this version would have an easier learning curve, which would enable wind and string players to master sight-reading more quickly.

2) The pure version discards the existing single and double sharp and flat symbols and replaces them with Sagittal single and double apotome symbols that mean exactly the same thing. Pure Sagittal symbols take up less space on a manuscript, thereby presenting a cleaner appearance. However, it is necessary to learn which symbols are *apotome complements*, i.e., pairs of symbols that, added together, equal an apotome. The pure version would probably be more easily read by keyboard players (once the symbols are learned), inasmuch as there would be less possibility for confusion or ambiguity in perceiving which symbols alter which notes, since there is a one-to-one relationship between the two.

It is possible for the two versions of the notation to coexist, with one or the other being preferred for different applications, or the mixed version might serve as an intermediate step toward the eventual exclusive adoption of the pure version. Whatever the case, having two versions should not produce very much confusion, since the single-shaft symbols have exactly the same meaning in both.

Bosanquet's Influence

The third idea to be incorporated into the notation is Bosanquet's use of sloping lines to indicate commatic alterations. This is illustrated in the top row of Figure 2 as it would be adapted for use in 72-ET, where the apotome (i.e., sharp or flat alteration) is equivalent to 6 syntonic or Didymus commas, 80:81, which we will refer to as the *5 comma* from this point on. These lines by themselves do not provide a foolproof way to distinguish up from down, but if either one of them is placed to the left of a vertical line, a half-arrow is formed, which clearly points in the appropriate direction, providing an unmistakable indication of its meaning. In the second row of Figure 2, the 5-comma down and up is represented by the two symbols immediately to the left and right of the natural sign. The 5-comma symbol-pair is basic to the Sagittal notation, being used for this single purpose in the overwhelming majority of equal temperaments (ET's), as well as for rational (or JI) notation. As stated previously, up-and-down opposites are not created by rotating a symbol by 180 degrees, but rather by mirroring it vertically.

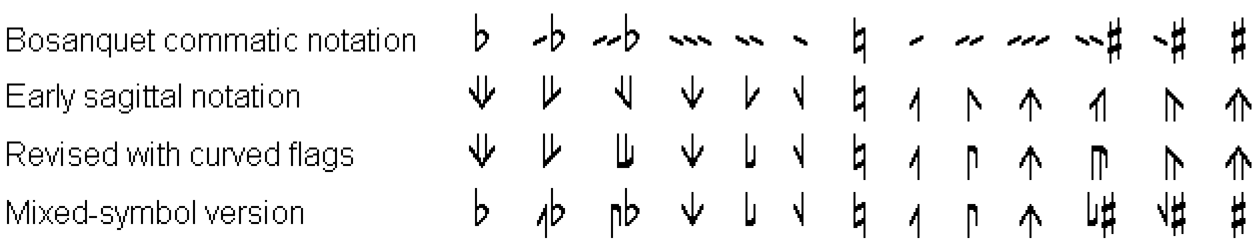

Figure 2 - Comparison of Sagittal Notation with Bosanquet Commatic Notation for 72-ET

When communicating about the notation via electronic mail, we use combinations of ASCII characters in text to simulate Sagittal symbols: the vertical bar (or "pipe") character "|" to represent an arrow shaft, a capital "X" to represent the crossed lines of the double apotome, and slash "/" and backslash "\" characters to represent halfarrowheads, which we call *flags*. Thus the 5-comma symbol that represents 1 degree of 72-ET (1º72) consists of a single arrow shaft and straight left flag, /|, while the 11-diesis symbol that represents 3º72 has a single shaft with straight left and right flags, /|\. A shorter name for a straight flag is a *barb*. Initially, the symbol for 2º72 was derived by subtracting the left barb from the 3º symbol, leaving a single shaft and right barb, |\. The other symbols follow logically, as shown in the second row of Figure 2.

Reducing Lateral Confusability

Due to the possibility of confusion between symbols that are lateral mirror images of one another, particularly when sight-reading at high speed, we decided that a different sort of flag should be used in place of a right barb |\ for 2º72, and we agreed on a single-shaft symbol with a convex curved right flag (shown in the third row of Figure 2), which we designated to represent a septimal or Archytas comma, 63:64. We will refer to this as the *7 comma* from this point on. A shorter name for a convex flag is an *arc*. The ASCII notation for an arc is a parenthesis that curves away from the vertical line character, so a 7-comma-up is |). For the ASCII simulation of downward pointing arrows, vertical lines are replaced with exclamation marks, and “X” is replaced with “Y”. We have also devised an ASCII *shorthand* for single-shaft symbols only, where each is represented by a single ASCII character. ASCII simulations of the symbol sequences in rows 3 and 4 of Figure 2 are shown in Figure 3.

<table>
<tr><td>\!!/</td><td>!!/</td><td>!!)</td><td>\!/</td><td>!)</td><td>\!</td><td>|//|</td><td>/|</td><td>|)</td><td>/|\</td><td>||)</td><td>||\</td><td>/||\</td></tr>
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<tr><td>b</td><td>b/</td><td>bf</td><td>v</td><td>t</td><td>\</td><td>e</td><td>/</td><td>f</td><td>^</td><td>#t</td><td>#\</td><td>#</td></tr>
</table>

Figure 3 - ASCII Simulation of Sagittal Symbols for 72-ET

Note that in the ASCII simulation the mixed version has the sharp or flat character to the *left* of the Sagittal symbol, so that it would immediately follow the letter indicating the letter nominal; e.g., F#\! would be the equivalent of F||\.

If a 5 comma (80:81, ~21.5¢) is added to a 7 comma (63:64, ~27.3¢), the result (arrived at by *multiplying* the ratios) is the 35 diesis, 35:36 (~48.8¢), which is represented by a symbol that combines the 5-comma and 7-comma flags: /|). In addition to representing this comma exactly, the same symbol can also be used to represent the *13 diesis* (1024:1053, ~48.3¢), which is almost the same size. We named the very small difference between these two intervals the *tridecimal schismina*[5], 4095:4096, ~0.42 cents. By allowing such small intervals to be ignored, it is possible to achieve a more economical use of symbols.

A convex left flag or arc (| was subsequently defined with the rather complex ratio 45056:45927 (~33.1¢) that, when added to the 7 comma, |), would result in (|) with ratio 704:729 (~60.4¢). This is the large 11 (abbreviated as 11L) diesis, which is useful as

[5] We adopted the term *schismina* to designate a very small class of intervals that are smaller than ~1.8 cents, in order to distinguish them from intervals such as the 19-schisma (512:513, ~3.38¢) and the 5-schisma (32768:32805, ~1.95¢), which are directly symbolized in the notation.

the apotome complement of the (medium-size) 11 diesis (abbreviated as 11M). In other words, /|\ plus (|) equals /||\ (a sharp, or apotome). Finally, if this left arc is combined with a right barb, the result is the large 35 (or 35L) diesis (|\, 8192:8505 (~64.9¢), which also differs from 26:27, the large 13 (or 13L) diesis (~65.3¢), by the tridecimal schismina. These symbols are also apotome complements, i.e., /|) plus (|\ equals /||\.

With these symbols (assuming that one is willing to ignore the tridecimal schismina), there are four different ways to notate 13-limit ratios that are in the neighborhood of a neutral third above C=1/1:

1) For 11/9, lower a Pythagorean E (81/64) by 704:729 giving E(!)
2) For 27/22, lower a Pythagorean E by 32:33 giving E\!/
3) For 16/13, lower a Pythagorean E by 1024:1053 giving E\!)
4) For 39/32, lower a Pythagorean E by 26:27 giving E(!/

Notice that the symbols for the 11 dieses have both left and right flags alike (either barbs or arcs), while those for the 13 dieses each have one barb and one arc. With the 13-diesis symbols the potential for lateral confusability enters the picture, but we have alleviated the problem somewhat by specifying that the large 13 diesis symbol should be noticeably wider than the (medium-size) 13 diesis symbol.

To provide symbols for primes above 13 (and to notate finer divisions of the octave) we devised two more types of flags that can be used on either the left or right side of an arrow shaft. These are physically smaller in size and are used to symbolize smaller commas than either the straight or convex flags. The *concave flag* consists of a concave curve or *scroll*, while a *wavy flag* consists of a concavoconvex curve or *boathook*. Left and right scrolls are simulated in ASCII by)| and |(, respectively, while boathooks are ~| and |~. Each Sagittal symbol thereby consists of either one or two (or in rare instances three) flags. Where there are two flags, they are most commonly on opposite sides. To minimize lateral confusability between symbols that have the same flags on opposite sides, we specified that there should be differences in the physical sizes of the left and right versions of the same flag type, according to the sizes of the commas they represent.

The Spartan Symbol Set

We tried many different combinations for matching scrolls and boathooks with the commas, kleismas, and schismas associated with the higher primes before arriving at our final selection. Although at times it seemed that there would be no end to our discussion of how many schisminas could vanish on the head of an arrow, we never considered the time wasted that was spent on ideas that were subsequently discarded, because having tried as many ways as we could possibly think of, we could be more confident that we had found the best.

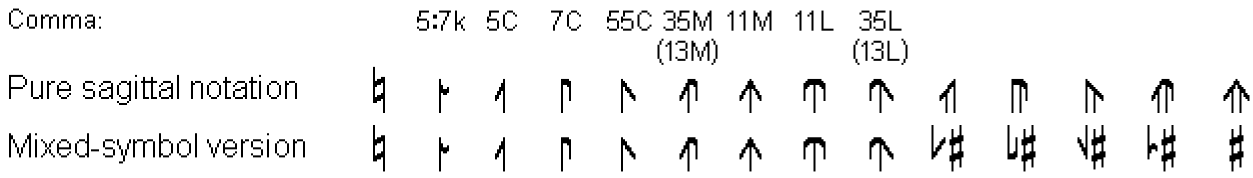

Figure 4 - The Spartan Symbol Set

Fortunately, many tunings will require only the *spartan* set of eight single-shaft symbol pairs, seven of which are composed of barbs and arcs alone (see Figure 4). With the spartan symbol set it is possible to notate all of the 9-limit consonances (exactly) and 16 harmonics and subharmonics (exactly, except for the 13th) in just intonation, plus over 40 equal temperaments (including most of the popular ones below 100 tones per octave).

It is therefore not necessary to learn very many of the symbols of the Sagittal notation in order to use it – only those required for a particular tuning or odd limit. Additional symbols may be learned as needed for more complicated tunings, higher odd limits, or special applications.

Symbol Definitions

Listed in Table 1 are some of the most commonly required single-shaft Sagittal symbols and the commas[6] they represent. The symbols that are used to notate prime harmonic factors relative to the natural Pythagorean notes are shown in **bold type**. The symbols that represent commas used to notate the primes from 5 through 29 each have a single flag, except for 11 and 13. The 11 and 13-defining intervals are dieses rather than commas, and their symbols each contain two flags (i.e., the sum of two commas), inasmuch as it is not economical to have a single flag representing an interval as large as a diesis. The right barb |\ is therefore defined as the 55 comma (54:55), i.e., the 11M diesis minus the 5 comma. We defined the right scroll |(as the 5:7 kleisma (5103:5120, the difference between the 7 and 5 commas, ~5.8¢), and we found that (by ignoring various schisminas) it could also serve as the 11:13 kleisma (351:352, the difference between the 11M and 13M dieses, ~4.9¢), or when added to the left boathook ~| that represents the 17 kleisma (2176:2187, ~8.7¢), it would result in a symbol ~|(that can represent the 17 comma (4096:4131, ~14.7¢).

Although there is not a complete one-to-one correspondence of symbol-elements (or flags) to primes such as was originally proposed, still only eight flags are required to notate the eight primes from 5 through 29. For some of these primes there are symbols for two different commas, which (among other things) allows the 17th, 19th, and 23rd harmonics to be notated as alterations to either sharped or flatted notes in order to

[6] The term “comma” is used here in a broad or generic sense, whereas in Table 1 it is also applied more specifically to intervals ranging from ~11.7 to ~33.4 cents, above which they are labeled as “dieses” (with large dieses exceeding 1/2-apotome, ~56.8 cents, and small dieses being those less than ~45.1¢). Intervals ranging from ~4.5 to ~11.7 cents are labeled as kleismas and those from ~1.8 to ~4.5 cents as schismas. Anything less than ~1.8 cents is a schismina. These boundaries were determined to be both useful and necessary to distinguish small ratios having the same combination of prime factors above 3.

Table 1- Selected Sagittal Symbols and Their Commas

Symbol & Comma Name		Comma Ratio & Size	Apotome Complement
	5 schisma	**32768:32805**, ~1.954¢	=
	19 schisma	**512:513**, ~3.378¢	=
	5:7 kleisma 11:13 kleisma	5103:5120, ~5.758¢ 351:352, ~4.925¢	=
	17 kleisma	**2176:2187**, ~8.730¢	=
	7:11 kleisma sum of flags	891:896, ~9.688¢ 189:190, ~9.136¢	=
	17 comma sum of flags	**4096:4131**, ~14.730¢ 119:120, ~14.487¢	=
	23 comma	**729:736**, ~16.544¢	=
	25 comma (diaschisma)	2025:2048, ~19.553¢	=
	19 comma sum of flags	**19456:19683**, ~20.082¢ 432:437, ~19.922¢	=
	5 comma	**80:81**, ~21.506¢	=
	7 comma	**63:64**, ~27.264¢	=
	55 comma	54:55, ~31.767¢	=
	7:11 comma 13:17 S-diesis **29 S-diesis**	45056:45927, ~33.148¢ 51:52, ~33.617¢ **256:261**, ~33.487¢	=
	5:11 S-diesis 7:13 S-diesis 11:17 S-diesis	44:45, ~38.906¢ 1664:1701, ~38.073¢ 1377:1408, ~38.543¢	=
	25 S-diesis 5:13 S-diesis	6400:6561, ~43.013¢ 39:40, ~43.831¢	=
	35 M-diesis **13 M-diesis** 125 M-diesis	35:36, ~48.770¢ **1024:1053**, ~48.348¢ 243:250, ~49.166¢	
	11 M-diesis	**32:33**, ~53.273¢	
	11 L-diesis	**704:729**, ~60.412¢	
	35 L-diesis **13 L-diesis** 125 L-diesis	8192:8505, ~64.915¢ **26:27**, ~65.337¢ 512000:531441, ~64.519¢	

provide alternate spellings for the intervals these make with the 12th, 16th, and 18th harmonics.

In reponse to a request for a means to distinguish between pitches differing by a 5-schisma (32768:32805, ~1.95¢), we introduced the *accent mark*, which may be added to the left of a Sagittal symbol to alter pitch by a 5-schisma in either the same or opposite direction of the symbol. Two accented symbols, along with their apotome complements (also accented) are included in Table 1 to illustrate this: an *acute* (or upward-sloping) accent mark for an upward alteration in pitch, and a *grave* (or downward-sloping) accent to indicate a downward alteration. These accent marks greatly increase not only the number of available symbols, but also the number of ratios that can be notated exactly. In addition, they allow for finer distinctions in pitch that: 1) enable those rational intervals not represented exactly, to be approximated with greater precision, and 2) make it possible to notate finer divisions of the octave.

Where more than one ratio (or role) is given for a symbol in the table, the first one listed is the *primary role* for that symbol, i.e., the ratio that *exactly defines* that symbol. Secondary comma roles are therefore only *approximated* by symbols in the notation of rational (or just) pitches or intervals and are valid in a given ET only if the difference between the ratios for the primary and secondary roles is a schismina that vanishes in that ET. Primary comma roles were chosen on the basis of which commas would be used to notate the most *popular* ratios, as determined from ratio occurrence statistics obtained by Manuel Op de Coul from the Scala archive of over 2000 historical and experimental tunings. As expected, ratio popularity was found to have a high inverse correlation with both prime limit and product complexity.[7]

A sequence of symbols for the simplest or most useful commas in Table 1 is shown in Figure 5, beginning with a natural and ending with a double sharp. Single-shaft symbols up to the 11M diesis are in the top row. The sequence continues from right to left in the second row (for both the mixed and pure versions), so that the symbols appear directly below those in the first row for which they are apotome complements. Notice that in the mixed version these are simply the symbols in the top row mirrored vertically and combined with a sharp. The symbols in the third and fourth rows correspond to those in the first and second rows respectively, with a sharp added.

The sequence of symbols progressing downward from a natural to a double flat would differ from these only in that each Sagittal symbol would be mirrored vertically and conventional sharp and double-sharp symbols would be replaced with flat and double-flat symbols, respectively. (Some of these appear below, in Figure 6.) We call this the *athenian*[8] symbol set.

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[7] The prime limit of a ratio is the largest prime factor that can be extracted from either side of the ratio in lowest terms. Similarly its odd limit is the largest odd factor. The product complexity of a ratio is obtained by multiplying together both sides of the ratio in lowest terms.

[8] Since we had already named one symbol set after Sparta, we thought it fitting to honor the other pre-eminent ancient Greek city-state with a symbol set name, particularly in light of its contribution to our musical heritage and its reputation as a philosophical forum (not unlike present-day Internet discussion groups).

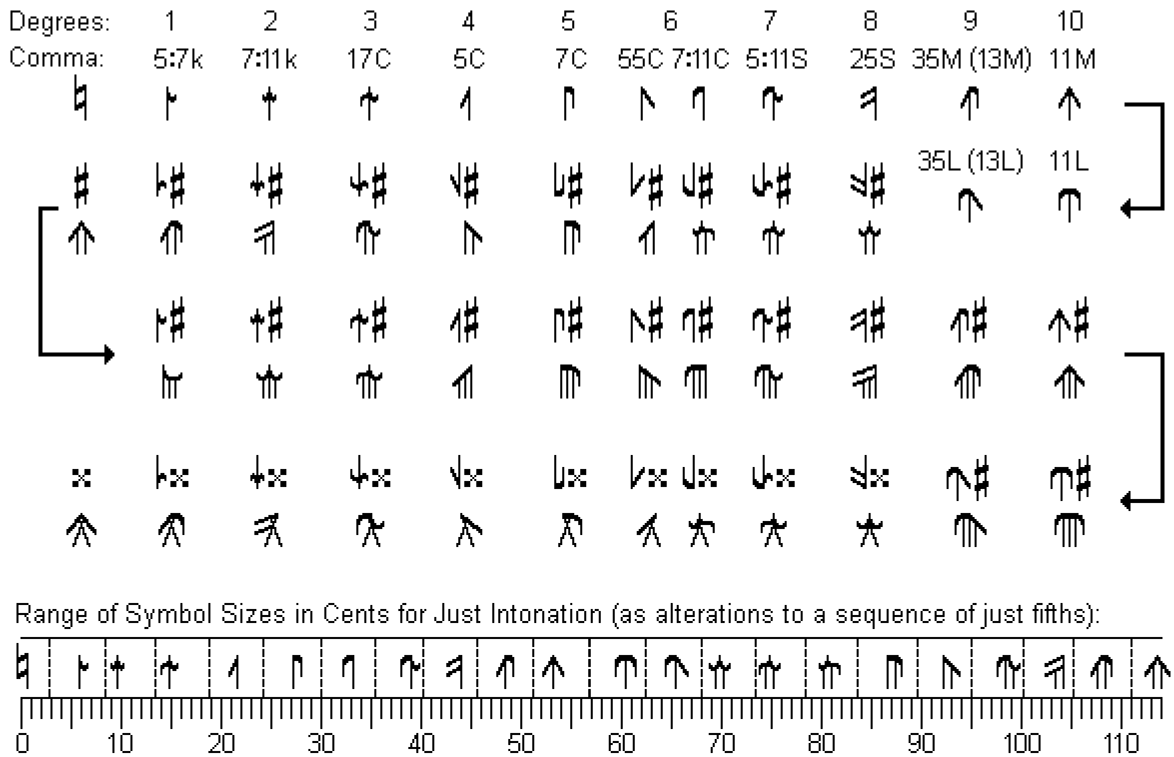


Figure 5 - Sequence of Athenian Sagittal Symbols

In Figure 5 there are two different symbols given for the 6th position in the sequence (not counting the natural sign). Using the 7:11-comma (| in preference to the 55-comma |\ for this position results in the medium-precision notation for just intonation (or rational pitches) which we refer to as *athenian JI notation*. (If the 55-comma is used instead, this then becomes the symbol set for 224-ET.[9]) When rational intervals or pitches must be *approximated* by symbols in athenian JI notation, the difference in cents between the ratio of the desired pitch and that of a nearby tone in a Pythagorean sequence containing 1/1 is calculated, and the appropriate symbol is determined by reference to a table of boundaries (easily accomplished with a computer spreadsheet or other software, such as Scala[10]), as illustrated in the diagram at the bottom of Figure 5. While it appears that pitch approximations could amount to as much as 3 cents, in actual practice approximations even as large as half this size rarely occur (due to the fact that the schisminas which define the semantics of the notation are very small), and

[9] Inasmuch as the 7:11 comma and 55 comma differ by less than 1.4 cents, either one would be suitable in this symbol sequence for notating either JI or 224-ET. The 7:11 comma symbol seems to be preferable for JI, because it *exactly* notates two 11-limit consonances (11/7 and 14/11), whereas the 55 comma symbol is generally preferable in an ET symbol set, because it results in matching sequences of flags (which are easier to remember) for the single- and double-shaft symbols in the pure version. However, it might make sense to use instead whichever symbol does not result in a laterally-confusable pair for the ET. We are still undecided on this issue.

[10] Version 2.2 or later, of the Scala software, which supports the Sagittal notation, is available from the Scala Home Page at http://www.xs4all.nl/~huygensf/scala/. However, at the time of writing there were still some issues to be resolved regarding the *preferred* notation of some pitches.

only then for ratios involving complex or unusual combinations of prime factors or less-preferred spellings. It is expected that the overwhelming majority of users will conclude that the athenian JI symbol set is quite acceptable for their purposes, with higher-precision symbol sets being necessary only for certain theoretical (or other special) applications.

A Musical Example

A musical example in just intonation using Sagittal notation appears in Figure 6 in both mixed and pure versions. This example can be heard in various tunings on the Sagittal website at http://users.bigpond.net.au/d.keenan/sagittal/exmp/. In the first measure the 5 comma symbol is used to make Pythagorean major thirds just; notice that the arrow-like symbol points downward and that the straight left flag also slopes downward.

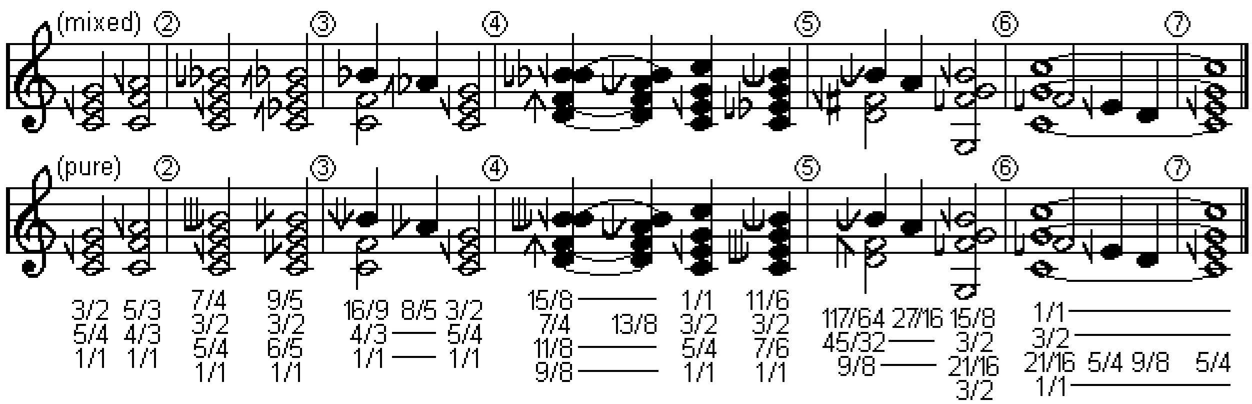


Figure 6 - A Musical Example in Sagittal Notation

In measures 2 and 3 there are three different minor sevenths: 1) 7/4 (a harmonic seventh) is a Pythagorean B-flat *lowered by a 7 comma*; 2) 9/5 (a large minor seventh) is a Pythagorean B-flat *raised by a 5 comma*; and 3) 16/9 is an *unaltered* Pythagorean B-flat. Observe the similarity in appearance between the two versions of the symbols, particularly the positive slope of the straight flags for the second chord of measure 2 and for 8/5 in the second beat of measure 3, which provides a clue that these are tones that are being *raised* by a 5 comma (even though the double-shaft symbols on the second staff are pointing *down!*).

Measure 4 uses some ratios of 11 and 13. Observe that different symbols (with like left and right flags) are used for 11/8 (up from F) and 11/6 (down from B), while 13/8 (the first *approximated* ratio in this example) has a symbol with one convex and one straight flag. Using single-shaft symbols for the 11L and 13L dieses in the mixed version makes it unnecessary to combine sharps or flats with 11M or 13M-diesis symbols that alter in the opposite direction, after the idea that it is simpler to read half of a flat than a whole flat less half.

The first chord in measure 4 illustrates a potential for confusion that arises if two notes of a chord are positioned on the same line or space. With the pure notation it should be clear that the left symbol alters the left note and the right symbol the right note (and if one of the notes is unaltered, then a natural sign could be used for clarity). However, with the mixed notation there are three symbols for two notes, which could tend to slow down the reading process. Moreover, if a situation is encountered where the first note is altered by only a single Sagittal symbol and the second by only a sharp or flat, a certain ambiguity occurs: do the two symbols together alter one or both notes, or does the left one alter one note and the right one the other?

This would seem to pose a strong argument in favor of adopting the pure version, at least for any music containing chords. But for orchestral parts the mixed version would not suffer from this problem, and symphonic players would more likely appreciate its gentler learning curve.

Measure 5 has a more complex ratio (117/64) for one of the notes, but examination of the symbol will reveal it to be a ratio of 13 (an approximated ratio 9:8 above the 13/8 in the previous measure). This measure also contains the only sharped note in the example (45/32); again note the significance of the downward slope of the straight flags in both versions.

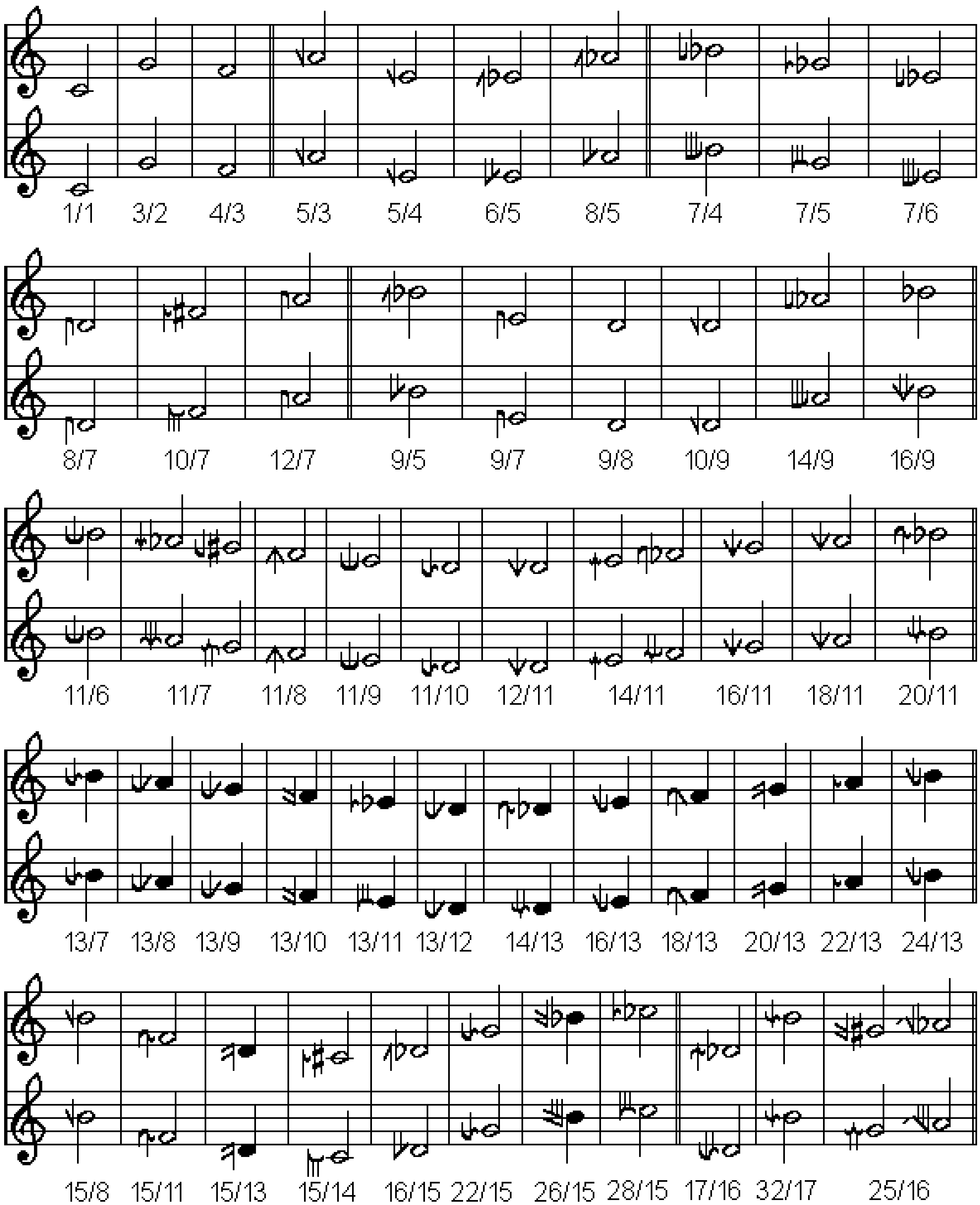
To aid in the determination of how the athenian symbols are used for just intonation, Figure 7 has been provided for reference. The upper staff shows each ratio in mixed-symbol notation, while the lower staff illustrates the pure Sagittal version. Quarter notes (crochets) are used to identify those instances where ratios are approximated by symbols, and three ratios (11/7, 14/11, and 25/16) are shown with alternate spellings. The exact alternate spelling for 25/16 is shown using a diaschisma symbol (to illustrate employment of an accented symbol), but athenian-level notation would dispense with the accent mark, thereby approximating this ratio using a 5-comma-down symbol.

A Common Notation for JI and ET's

This section heading, which has served as the subject line in most of the correspondence for our notation development project, asserts that the Sagittal notation can be used for both just intonation and equal divisions of the octave, but so far we have described only how Sagittal symbols are used to notate ratios. As was previously stated, for an ET the nominals are in a chain of the best fifth of that ET, but what about the accidentals? A first pass at notating ET's is to specify that the best approximation to any ratio should be notated in the same way as that ratio. However, this by itself is insufficient to determine which symbol to use for a given number of degrees of a particular ET, since in many instances there are two or more ratios (requiring different symbols) that are represented.

Theoretically, every Sagittal symbol is meaningful in any equal division of the octave, so any symbol could theoretically be used in one way or another to notate any ET, as long as its primary comma role is a positive number of degrees in that ET. This makes it possible under certain circumstances to sight-read a composition written for a larger ET or in just intonation on an instrument designed for or tuned to a different ET, so long as the player is aware of which commas vanish in which ET. For example, 5-comma symbols would be ignored when translating just intonation into 19-ET and 31-ET, while 7-comma symbols would be ignored when translating into 22-ET. Since this requires a

Figure 7 - Sagittal Athenian Notation for Just Intonation



knowledge of musical acoustics that the average musician cannot be expected to possess, the limitations for doing this reside with the player more than with the notation.

It is desirable that players should not have to cope with different sets of symbols for the same ET in different compositions, so we have specified a *standard set* of symbols to be used for each ET. These have been selected using several criteria, including a symbol's prime factor limit, the division's prime number errors and consistency, and validity of secondary comma roles. Consistency of symbol flag arithmetic for the single-shaft symbols has been strictly maintained, but occasional inconsistencies have been allowed for double-shaft apotome complements in cases where they are not likely to be noticed.

To specify the symbol sets for the pure version of the notation, it is necessary only to list the symbols used to alter consecutive degrees up to the apotome. In every case, double-shaft symbols are determined by using (in reverse order, beginning at the apotome) the apotome-complements of the single-shaft symbols (as shown in Figures 5 and 13). Beyond this point the symbols are re-used beginning from the left, replacing a single shaft with a triple and a double shaft with an "X" (in the same way as shown for the athenian JI symbols in Figure 5).

For the mixed-symbol version, only the single-shaft symbols are used, either alone or in combination with conventional single and double sharp and flat symbols.

Listed in Figure 8 are the divisions that require no symbols other than those in the spartan set. In some of these divisions the fifths have such large errors that the divisions are not 1,3,9-consistent. In such cases it may be preferable to notate these as subsets of larger divisions, even though their native-fifth notations generally require fewer symbols.

Listed in Figure 9 are some divisions for which the notation is more complicated. Most of these require symbols outside the spartan set (e.g., those with scrolls and/or boathooks).[11] Several of these divisions (111, 152, 176, 183, and 224) have sets of single-shaft symbols such that the flags in the entire sequence of double-shaft symbols match those in a portion of the single-shaft symbol sequence, which makes it much easier to remember the symbol sequences.

It needs to be emphasized that the comma sizes in cents, as listed for the various symbols in Table 1, apply only to rational or JI tunings, in which the symbols alter tones in a chain of Pythagorean fifths. In various ET's these commas are represented by intervals that may be either larger or smaller than their rational sizes, or they may vanish altogether. This sometimes results in reversals in the size-order of the symbols, but in selecting the standard sets for the ET's we have sought to minimize this.

[11] The notation for 58-ET has not yet been finalized. Using the 55-comma symbol for 2º (along with its complement for 4º) is a simplification that would require only spartan-level symbols and would also result in a matching symbol sequence for the double-shaft symbols, whereas the athenian-level notation shown in Figure 9 eliminates the lateral confusability between the 5-comma /| and 55-comma |\ symbols.

In addition, Sagittal notations for some ET's not given in this paper may be found in the Scala software, available from the Scala Home Page at http://www.xs4all.nl/~huygensf/scala/.

Figure 8
Equal Temperaments Notated Using the Spartan Symbol Set

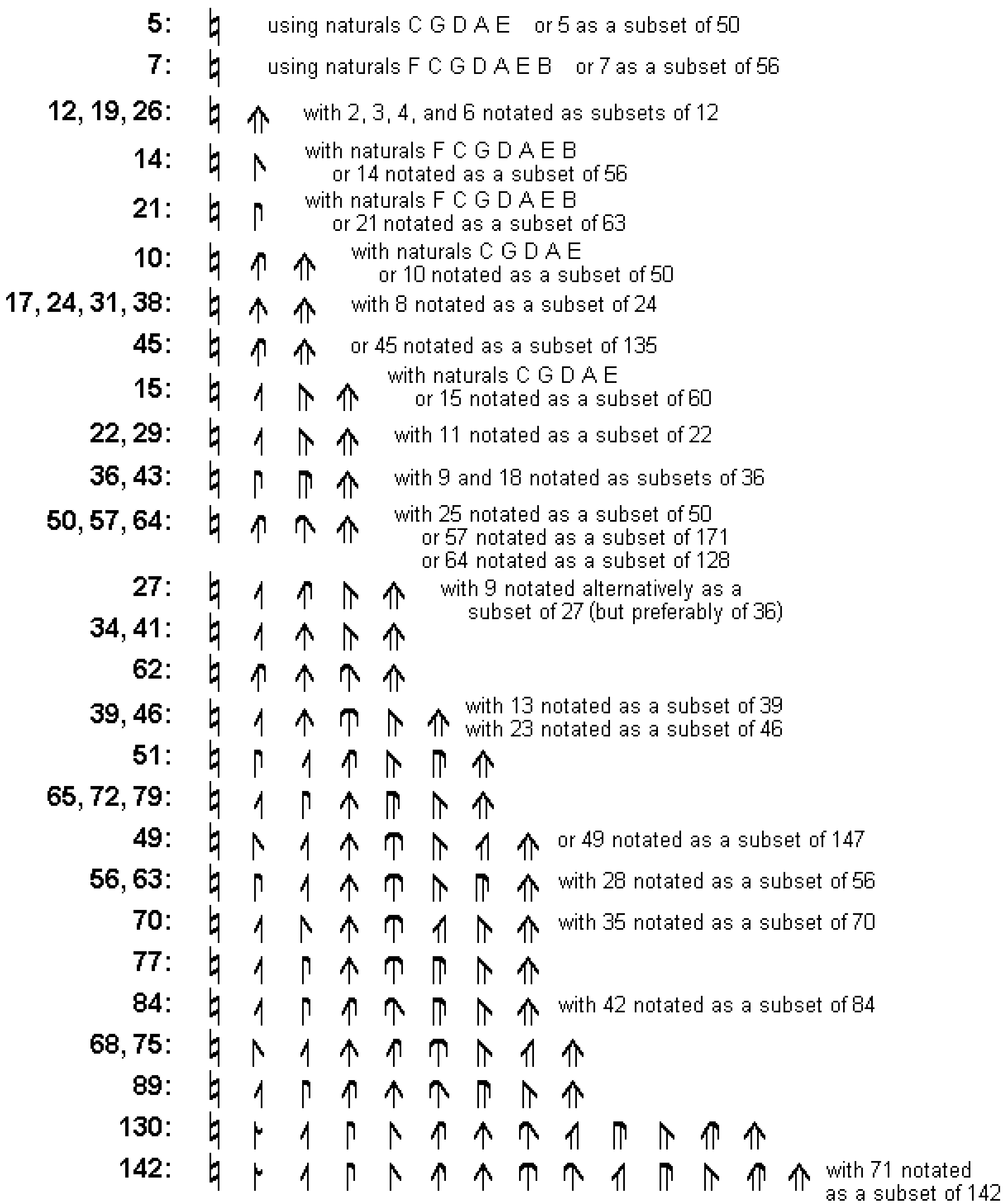
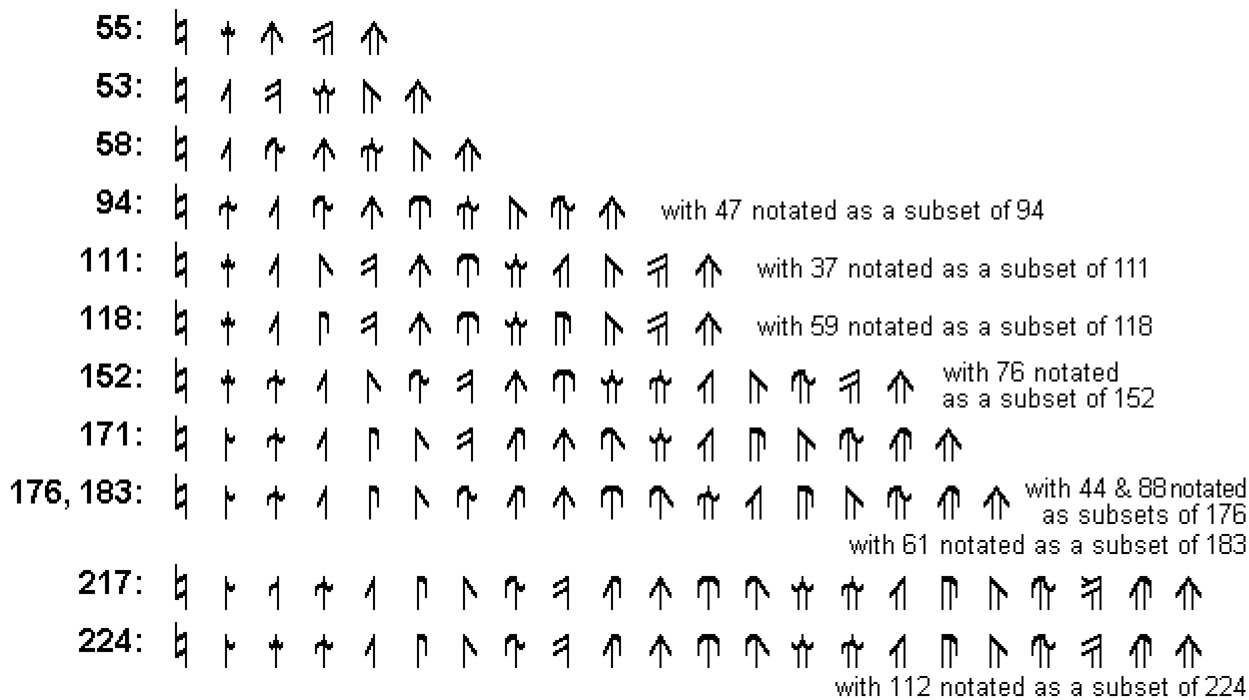


Figure 9
Symbol Sequences for Some Other Equal Temperaments

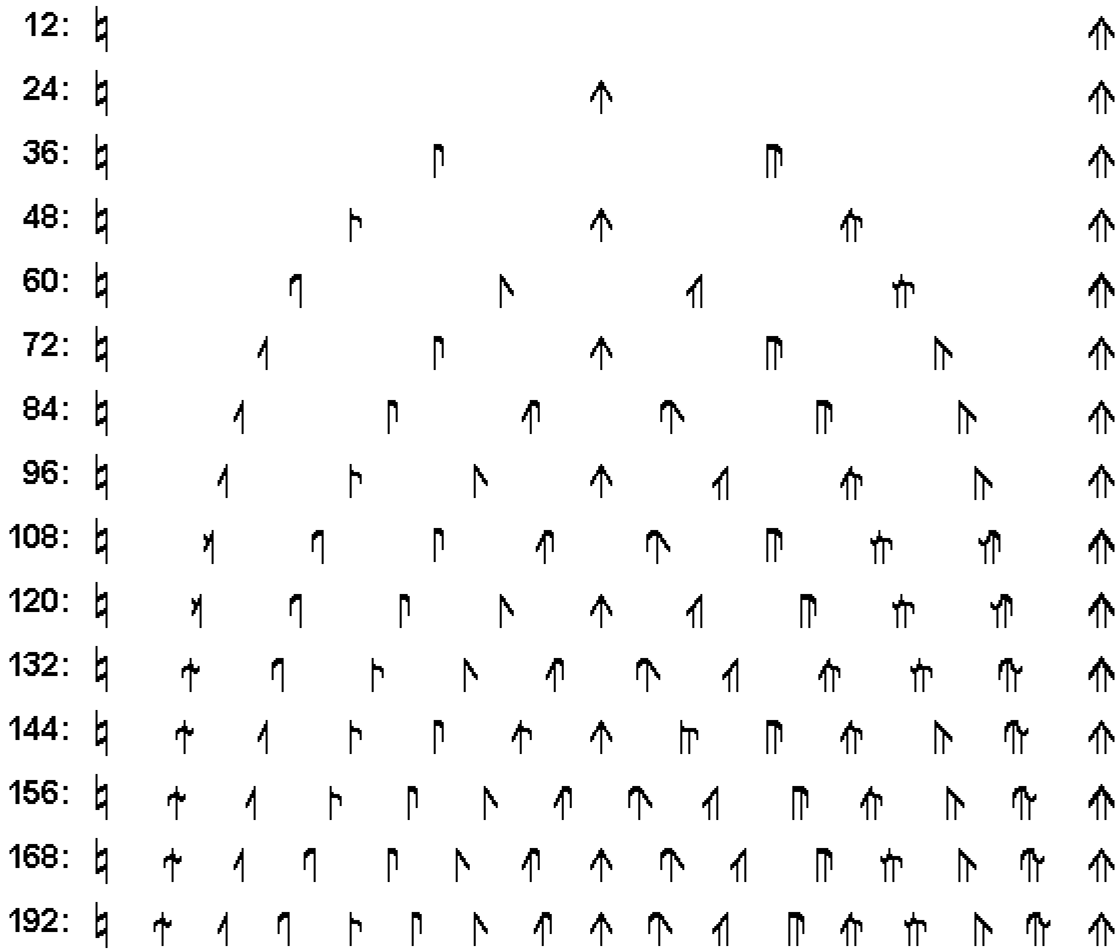


No microtonal notation system could be considered universal if it did not have the desirable properties that Gardner Read describes in his book *20th Century Microtonal Notation*. In particular it must cater to those who prefer to notate tunings relative to 12-tone equal temperament. In this application it must provide unique and consistent symbols for various fractions of the equal-tempered semitone. A set of symbols has been selected for this purpose, and although it adheres faithfully to the Sagittal principle of notating according to the best approximations of just or rational pitches, one may freely ignore this fact and simply consider the symbols as representing particular fractions of a semitone or tone. We call this the *12R* (12-relative) or *trojan*[12] symbol set, and it is shown in Figure 10. Note that most trojan symbols are also in the athenian set.

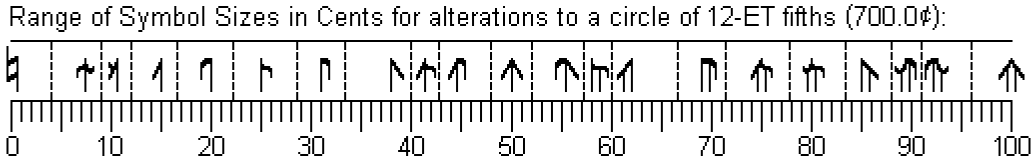
Each symbol in this set has been assigned a size range for modification of tones relative to a 12-ET circle of fifths (as indicated by the diagram near the bottom of Figure 10). The most obvious application of this is to ET's which are multiples of 12. These are shown in the top part of Figure 10. Observe that in some cases the same symbol is used for several slightly different fractions of a tone; for example the left barb is used for 1/12, 1/14 and 1/16 of a tone. But in no case does the size for a symbol in any of these ET's vary by more than ±2 cents, thus allowing a player who must read

[12] Since many musicians who use microtonal subdivisions of 12-equal have little or no desire to be involved with ratios, we thought it fitting to name this symbol set "trojan". This is in keeping with our "ancient city-states" theme, whereby we would be making available to them a gift-horse notation based on the rational numbers of JI. However, we need not fear that the ratios will escape and overrun the city; instead they will merely offer to provide a JI interpreting service, should this ever be required.

Figure 10
Trojan Symbol Sequences for 12R Equal Temperaments



Range of Symbol Sizes in Cents for alterations to a circle of 12-ET fifths (700.0¢):



2, 3, 4, and 6 notated as subsets of 12
8 notated as a subset of 24
9 and 18 notated as subsets of 36
16 notated as a subset of 48
20 and 30 notated as subsets of 60
42 notated as a subset of 84
32 notated as a subset of 96
54 notated as a subset of 108
52 notated as a subset of 156
66 notated as a subset of 132

parts in multiple tunings to develop an instrumental technique that uses nominal symbol sizes to arrive at approximate pitches, which may then be fine-tuned by ear. Again, it must be emphasized that these nominal symbol sizes in cents are valid only when the symbols are used to alter tones in a 12-ET circle of fifths.

These three lists of ET's are by no means exhaustive. We have also notated even higher divisions such as 270, 282, 306, 311, 342, 388, 494, 612, and beyond, as well as many below 224. At the present time these and other details involving the advanced features of the notation are still being worked out, including comma definitions for accented symbols and methods for notating linear, planar, and non-octave temperaments. Our selection of standard symbols for many divisions is still under review and subject to change, so they have not been listed. Since we have followed the principle that the overall complexity of the complete notation system should not make the simpler things more complicated, none of these developments will result in a significant change to anything presented here.

While the notation of linear or higher-dimensional temperaments has not yet been investigated in great detail, it has become evident that such a temperament may be readily notated using the notation for an ET that closely approximates it. Meantone temperament may therefore be treated as if it were a subset of 31-ET, while the Miracle temperament may be notated like 72-ET. For an irregular set of tones which are not described by ratios, it would be necessary to find some suitable division of the octave into which they could be consistently mapped, and the symbols for that division could then serve as the notation for that set, even if the set itself does not contain octaves.

Adaptive Just Intonation

One planar temperament that deserves special mention in this discussion is that having generating intervals of 2:3 and $(80:81)^{1/4}$ – the just fifth and the quarter-comma of meantone temperament. By referring to Figure 5, one may observe that, since the 5-comma is 4 steps in the athenian JI symbol set, these symbols may then be defined to represent *exact quarter-commas* in this temperament and may thereby be used to notate adaptive just intonation in which the pitches depart from a Pythagorean tuning by multiples of 1/4-comma.

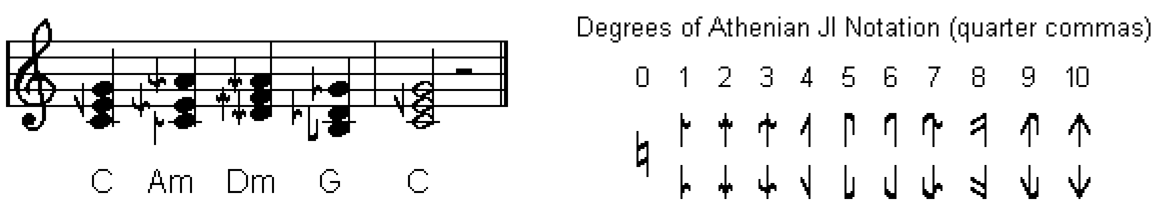


Figure 11 - 1/4-Comma Adaptive Just Intonation in Sagittal Notation

An example of this is given in Figure 11, which illustrates a simple chord progression (a "comma pump") in adaptive just intonation. Comma drift is eliminated by causing each repeated note to shift upward in pitch by 1/4 comma. As a result, each harmonic interval (or vertical sonority) is an exact 5-limit ratio, with only melodic intervals occurring as irrational ratios.[13] With repeated notes this very small change in pitch is virtually inaudible, but in instances where a note is held over between chords, the best effect is achieved by executing a rapid upward glide (rather than an instantaneous change) in pitch.

Some Purely Practical Considerations

What about existing notations? There is one for 72-ET devised by Ezra Sims in the 1970s with which many instrumentalists are already familiar. Should they be compelled to learn "yet another notation?" We think not. Good microtonal players are not abundant, and if they are not willing to learn a new notation, then that is their prerogative.

However, we don't think that this is an insurmountable problem. Now that we can have computer-generated instrumental parts, it would also be possible to have software that could translate one notation into another and then print any given part in whatever notation a player is comfortable with.

But there is more to be taken into consideration. Players fluent with the Sims notation must realize that if they are asked to play in systems other than 72-ET, then there is a real possibility that they would be required to read another notation – and for another tuning, yet another notation, at which point the real issue becomes clear: It is not so much having to learn a new notation that is the problem; rather it is discovering that the new notation that one has learned is not the end of the matter. And this will always be the case with *specialized* notations. While they each have their place and purpose, by their very nature they impose obstacles for composers who are attempting to get performances of works in different tunings.

There is a notation devised by Johnny Reinhard that would seem to overcome this problem in that it is not specific to any particular tuning. It is extremely easy to comprehend, and he claims that excellent results can be obtained with it. It consists of using 24-ET notation (a variation of Tartini's symbols) with numbers (i.e., signed integers) written above each note to indicate the number of cents by which that note should be altered to obtain the desired pitch. In essence, this is a notation for 1200-ET that can approximate any pitch of any tuning to within half a cent and any interval to within one cent.[14]

But is this the *generalized* notation that we are seeking? While it is highly successful for the employment of extended techniques for altering pitch on conventional (flexible-pitch) instruments, the employment of cents numbers is not particularly useful for instruments of *fixed pitch* that have been specially built for new tunings, e.g., a refretted guitar, a xylophone or metallophone, or a synthesizer operated by a generalized keyboard controller. Should wind instruments for another division (or better yet, multi-system instruments) be developed, then increments of 1200-ET are not going to be very

[13] This is further explained in the "adaptive JI" article in Joe Monzo's *Tonalsoft Encyclopaedia of Tuning* at http://www.tonalsoft.com/enc/.

[14] This notation is employed extensively for performances produced by the American Festival of Microtonal Music, of which Johnny Reinhard is director; see http://www.echonyc.com/~jhhl/AFMM/.

helpful. It is arguable that, without special wind instruments for other divisions of the octave, microtonality would be unattainable except by those virtuoso wind players who are able to master extended techniques for altering pitch, thereby putting off any possibility that it might one day become a part of the musical mainstream.[15] It should be clear that a generalized notation must not only be useful for all tunings, but also for all types of instruments, whether designed and built for 12-ET or for alternate tunings.

In the course of the notation's development, we discussed a few other practical matters that should be mentioned:

1) The Sims notation confronts the reality of reading instrumental parts under less than optimal conditions (such as poor lighting and the greater reading distance required when two players share a music stand) by having symbols that can be easily distinguished under these circumstances. While the symbols of the Sagittal 72-ET notation are quite distinct, tunings using more symbol pairs may require the use of slightly larger staves than usual for the parts in order to make them more readily distinguishable under adverse conditions.

2) An accommodation will be required for musicians who presently read the Sims notation, but are put off from learning the Sagittal notation because of the conflict in meaning between the virtually identical Sagittal 5-comma symbol pair (for 1º72) and the Sims symbol pair for 2º72. Ervin M. Wilson devised a modification to Bosanquet's sloping comma-lines that easily distinguishes the comma-up from the comma-down symbol by putting a vertical line through it. This causes it to resemble a "plus" sign, while leaving the comma-down symbol as is (resembling a "minus" sign). If the Sagittal 5-comma pair is replaced with the Wilson 5-comma pair, the result is the Sagittal-Wilson symbol set for 72-ET. This is shown in the third row of Figure 12, where the horizontal strokes of the Wilson 5-comma symbol pair have been slanted and broadened in accordance with the

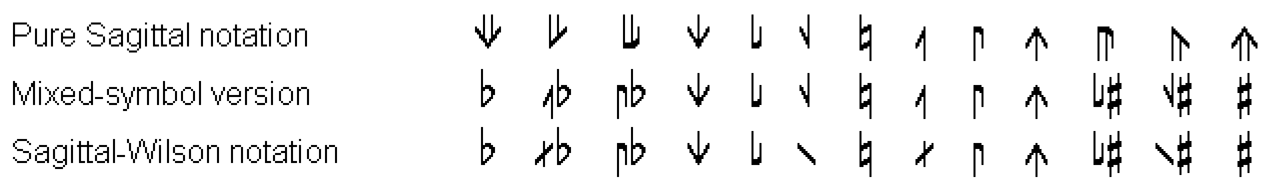


Figure 12 - Comparison of Sagittal-Wilson Notation with Pure and Mixed Sagittal for 72-ET

[15] Extended techniques require alterations in pitch (or "pitch-bending") of up to ±50 cents using conventional instruments, which generally requires a great amount of skill in order to achieve both accurate pitch and a reasonably good tone. Wind instruments built for 31-ET, for example, would allow musicians of lesser ability to play music written not only for that division, but also for 217-ET. Since 31 is a subset of 217, all of the pitches of 217 could be produced on a 31-tone instrument by pitch-bending in increments of ~5.5 cents (1º217) up or down, up to a maximum of around ±17 cents (i.e., less than a 5-comma, or hardly any more than what is required in 12-ET to play diatonic music in tune). Similarly, instruments built in 19-ET or 38-ET could make multiples of those divisions (i.e., 152, 171, 190, and 494) attainable. Any of these larger divisions could also be used as approximations to just intonation.

recommendations of Daniel Wolf.[16] The Sagittal-Wilson symbol set thus provides an intermediate step in a possible transition from Sims to Sagittal, with the potential for confusion minimized at each step in the process.

3) The excellent results achieved using the Reinhard method of notation for conventional instruments cannot be summarily dismissed. We have therefore concluded that it could be useful for a string, wind, or voice part in Sagittal notation to have the cents deviation from a 12-ET pitch[17] placed above the notes for those players who are able to benefit from this. In combination, these two approaches to notation would supply the same information in different formats that effectively complement one another. Given sufficient time a player could commit the cents deviations to memory for a frequently-played tuning and would then be able to rely mostly on the Sagittal symbols. Should instruments designed for microtonality subsequently become available, these parts would already be in a notation that would be meaningful to players using the new instruments.

   It should be observed that only with divisions of the octave that are multiples of 12 will there be fixed deviations from 12-ET pitches for the various Sagittal symbols, for it is only in these divisions that the symbols are indicating alterations to tones relative to a circle of fifths of exactly 700 cents. Even within this group of divisions a given symbol may differ slightly in size; e.g., the 5 comma is ~16.7 cents in 72-ET, but it is ~14.3 cents in 84-ET and 12.5 cents in 96-ET. And outside this group the size of the 5 comma (relative to a system's best fifth) can get considerably larger, e.g., ~54.5 cents in 22-ET. The important thing to keep in mind is that Sagittal symbols indicate *harmonic relationships* relative to a chain of fifths rather than fixed melodic intervals.

4) Since the Sagittal notation symbolizes intervals that are allowed to vary in size to accommodate many different tunings, it is necessary that sufficient information be provided in a score to specify the particular tuning that is intended, along with a pitch reference. It is expected that this information could be displayed in a standard format that will eventually be defined as a part of the formal specifications of this notation.

[16] See the "HEWM" entry at http://www.tonalsoft.com/enc/ beginning at the heading, "Daniel Wolf's version of HEWM" approximately halfway through the article. Since the Sagittal and HEWM notations both use the same up- and down-arrow pair for the 11 diesis, and since the Sagittal 7-comma symbol pair is somewhat similar in appearance to the HEWM 7-comma pair (particularly the down symbol), it would not be unreasonable to consider the Sagittal-Wilson symbol set as a dialect of HEWM notation.

[17] While the Reinhard method makes use of numbers of cents modifying 24-ET pitches, it is not possible to use separate quartertone symbols at the same time as Sagittal symbols, so the numbers would need to be specified relative to steps of 12-ET. The numbers would therefore be wider in range, but would never require more than 2 digits.

The *Sagittal* TrueType Font and Website

A scalable music font containing the Sagittal symbols (including the Wilson 5-comma symbol pair) is available free of charge at the Sagittal website, http://users.bigpond.net.au/d.keenan/sagittal/. This can be used with existing (and future) computer software products to generate high-quality music manuscripts in Sagittal notation. Also to be found at the Sagittal website are a character map for the font summarizing the uses of the various symbols, a set of scripts written by Jacob Barton to facilitate its use with the notation software *Sibelius*, and an entertaining "history" of the development of the Sagittal notation.

Engineered Evolution

In the early stages of its development, we decided to make the notation as versatile as we possibly could, because we recognized that if we took the time and effort to provide as much capability as possible from the start and to do it right, then it would be unlikely to need changing later (something that might upset those who had already started using it). Keeping this objective in mind, we devised specific symbols that could be used for many different tunings, both rational (JI) and tempered, making occasional modifications both to our choice of symbols that would be included in the Sagittal superset and to the exact definitions of those symbols.

Over the course of several years we took the notation through at least a half-dozen generations of development that, were it to occur by an evolutionary process involving actual microtonal practice, would probably require many centuries to arrive at. Sagittal notation could thus be described as a product of engineered evolution, but despite our best efforts we may well find that improvements are still possible, as it continues to be put to the test in actual use by others.

We eventually closed on a set of 31 single-shaft symbols (not counting accents, and counting vertically mirrored pairs as one symbol; see Figure 13) that can notate most ET's of interest, some having as many as 400 tones in the octave, and all rational intervals to within about two cents. Rational intervals are notated *without reference to any temperament*, and all of the most common ones are notated *exactly*, so it is only the more complex (or less common) ratios that must be approximated by reusing symbols for simpler ratios.

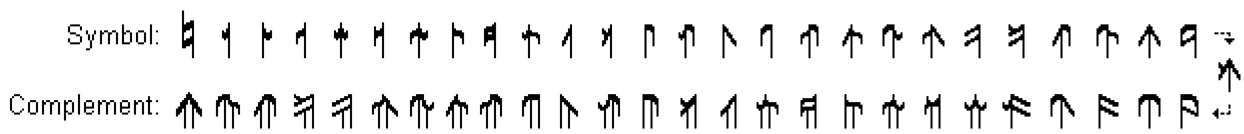


Figure 13 - Complete Symbol Superset from Natural to Apotome (Sharp)

In response to a subsequent request for a means of distinguishing between pitches differing by the tridecimal schismina (4095:4096, ~0.42¢) in electronic music applications, we also introduced additional accent marks that can be added to the *right side* of a Sagittal symbol (either left-accented or unaccented) to indicate an additional alteration, either up (acute) or down (grave), and either singly right-accented (to alter pitch exactly or approximately by a tridecimal schismina) or doubly right-accented (to alter pitch exactly or approximately by 2079:2080, ~0.83¢). These allow an even greater number of ratios to be notated exactly, while making it possible to notate divisions of the octave as high as 2460-ET.

In accordance with this, we have also defined symbol sets to be used for several levels of precision in just intonation. While medium-precision (athenian-level) JI, which uses twelve pairs of single-shaft symbols (without any accent marks), should be adequate for most purposes, we have also defined high-precision (in versions with or without left accents) and extreme-precision (utilizing both left and right accents) symbol sets for those with more exacting requirements, in which case a trade-off of simplicity for precision must be taken into consideration.[18] Our objective is that, in order for a notation system to be acceptable to as many composers as possible, it must not only be able to notate just about anything that anyone might want to do, but also just about anything that anyone *thinks that they might want to do*. Though some of these advanced features of the Sagittal system would never be used by most composers, it is reassuring to know that they are available, should they ever be needed.

While a casual first glance at the array of Sagittal symbols shown in Figure 13 might appear as bewildering as Chinese characters (or about what one might expect a 28th-century microtonal notation to look like), we can take comfort in the fact that most applications will use only a small fraction of these. As we have already seen, the most popular tunings require only the 3 single-shaft symbol pairs of the 72-ET set, many other tunings require only the additional 5 pairs of the spartan set, and practically everything else can be done with the additional 5 pairs of the athenian set.

[18] There is no formal symbol set for a "low-precision" Sagittal JI, but this could be achieved simply by mapping rational intervals to some particular division of the octave that is consistent to the required odd factor limit, e.g., to 72-ET for 11-limit JI or to 130-ET for 15-limit JI; both of these divisions use a subset of the relatively simple "spartan" symbol set. For comparison, the medium-precision (or "athenian") JI set referred to in the text has a pitch resolution comparable to 224-ET (in which the 5-schisma vanishes).

Conclusion

We believe we have described a *generalized* notation system that can handle practically any tuning that anyone would ever want to use. It would be meaningful for both conventional instruments and those specially constructed for alternate tunings, as well as for the strings and voice. We believe the depth and breadth of sources and experience drawn upon and the level of consensus reached, not to mention the sheer number of hours expended, are unprecedented in any previous microtonal notation design effort. We believe the result is backward compatible, logically consistent, visually coherent, aesthetically beautiful, unified, intuitive, flexible, practical, readable, simple, portable, universal, and extensible. And we envision a future when all microtonal musicians, whether composers, performers or theorists, whatever their instrument or musical style, will share a single harmonically-based *lingua franca* of pitch, no matter whether their tuning is justly intoned, equally tempered, or on one of the many middle-paths.

Acknowledgements

In closing, we wish to thank the following people for their comments, suggestions, and/or encouragement during the development of the Sagittal notation system thus far, (in alphabetical order): Yahya Abdal-Aziz, Jacob Barton, Gabor Bernath, Graham Breed, Paul Erlich, Mark Gould, Kraig Grady, Aaron Hunt, Marc Jones, Carl Lumma, Herman Miller, Alison Monteith, Joe Monzo, Ted Mook, Manuel Op de Coul, Joseph Pehrson, Johnny Reinhard, Prent Rodgers, Joel Rodrigues, Klaus Schmirler, Margo Schulter, Samara Secor, Gene Ward Smith, Dan Stearns, Jon Szanto, Robert Walker, Robert Wendell, Danny Wier, and Ozan Yarman. But we take full responsibility for whatever shortcomings may remain in the notation, since the final decisions were ours.

We would also like to thank the following, who didn't know that they were contributing to Sagittal at the time (in chronological order of their contribution): Pythagoras of Samos, Archytas of Tarentum, Didymus the Musician, Claudius Ptolemy, Giuseppe Tartini, Robert H. M. Bosanquet, Hermann von Helmholtz, Alexander Ellis, Ervin M. Wilson, Ivor Darreg, Ezra Sims, Gardner Read, Paul Rapoport, and Daniel Wolf.

Without the collective insight and experience of these two groups of individuals, the development of the Sagittal notation system would not have been possible.

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