TEMPERAMENT; OR, THE DIVISION OF THE OCTAVE.

By R. H. M. Bosanquet, M.A., F.R.A.S., F.C.S., Fellow of St. John's College, Oxford.

PART I.

1. Introduction.

The investigations which form the subject of the present communication had their origin in a practical source. It is conceived important that this should be understood, as many musicians regard the problem as a purely theoretical one, undeserving of practical attention. It was in taking part in the tuning of an organ that the effect of the ordinary equal temperament was first realised by the writer as a matter affecting musical sounds in practice, as distinguished from theory. It is the writer's experience that after the ear has once been attracted to this effect, it never fails to perceive it in the tones of instruments tuned in the ordinary way. Some tones show the effect more, some show it less. Among keyed instruments the worst effects are produced by the ordinary harmonium; next to this comes the full-toned modern grand piano, which is often unpleasant in slow harmony, especially if, as is generally the case, the temperament is not very uniform; * and the best tones, which show

* The chord of E major is constantly found 'rank' on these pianos. This would suggest some defect in the process of tuning ordinarily used.

the effect least, are those of soft-toned pianos with little power, and the ordinary organ diapason-stops—especially those old-fashioned, sweet-toned diapasons, which are rapidly disappearing before the organ-builders of the present day. The constant perception of these effects served as an inducement to a study of the subject, pursued in the first instance with the practical object of the improvement of instruments, and afterwards also for the sake of the interest attaching to the theory developed.

The problem to be dealt with is well stated as follows, in the Preface to 'A Theory of Harmony, founded on the Tempered Scale,' by Dr. Stainer:—'When musical mathematicians shall have agreed amongst themselves upon the exact number of divisions necessary in the octave; when mechanists shall have constructed instruments upon which the new scale can be played; when practical musicians shall have framed a new notation, which shall point out to the performer the ratio of the note he is to sound to the generator; when genius shall have used all this new material to the glory of art—then it will be time enough to found a new theory of harmony on a mathematical basis.'

This passage was of considerable use in directing attention to the points of importance in theory; it contains, however, some confusion of ideas, as will be pointed out immediately.

In the first place, before any conclusion can be come to as to the number of divisions necessary in the octave, it is clear that the theory of the division of the octave must be studied in a more complete and comprehensive manner than has been usual in the theory of music. In fact, when we come to examine the subject, we shall find that, although the properties of isolated systems have been studied here and there, yet no comprehensive method has been given for the derivation and treatment of such systems;* and the establishment of such a method will be the first point which will demand our attention. We shall then come to the conclusion that different systems have their different advantages; and we may contemplate the possibility of music being written for, or adapted to, one system or another, just as hitherto music has been written for performance in one key or another of the equal temperament.

In the second place, notation will be provided, by which the exact note intended to be played can be indicated to the performer, in those systems in which a modification of the ordinary notation is necessary. The notation is so constructed as to supplement the ordinary notation without altering it, and the signs required in addition to the ordinary notation are few in number and simple in their character.

Again, the problem of instruments has been solved in a general manner as far as keyed instruments are concerned. A generalised keyboard has been devised, by means of which it is

* See, however, Mr. A. J. Ellis on the 'Temperament of Musical Instruments with Fixed Tones.'—Royal Society's Proceedings, 1864.

possible to control the notes of all systems which proceed by continuous series of equal fifths; and this keyboard has been constructed and applied.

The confusion of ideas above alluded to arises from the assumption that the theory to be employed will be based on the derivation of scales from some one harmonic series, as well as from the division of the octave. Now this will never be true. If scales are derived from the division of the octave, their notes can never be more than approximations to the notes of any one harmonic series. In some of the systems subsequently developed, conditions having reference to the properties of harmonics will be employed; for example, we may make our fifths or our thirds perfect. This class of conditions is regarded as being derived from the harmonic series of each pair of notes employed. But the notation employed has not in any case reference to the ratio of the note sounded to any generator.

In the present introductory paper, the principal properties of the class of systems dealt with will be established in a general manner, the notation above referred to will be explained, and a brief account will be given of what may be called the 'principle of symmetrical arrangement.' This principle is the foundation of the arrangement of the keyboard above referred to; and its chief characteristic is, that any given interval, or combination of intervals, presents the same form on such an arrangement, on whatever notes it is taken—whence the form of fingering on the keyboard is the same in all keys.

2. Expression of Intervals.

Before commencing the treatment of these subjects, it will, however, be necessary to make some remarks on the method employed for the expression and calculation of intervals.

All intervals will be expressed in terms of equal temperament semitones. The letters E. T. will be used as an abbreviation for the words 'equal temperament.' Thus an octave, which is 12 E. T. semitones, will be written as 12; the 53rd part of an octave will be written as $\frac{12}{53}$, or ·22642. Five places of decimals will be generally considered sufficient.

The following rules for transforming vibrations ratios into the equivalent E. T. interval can be made use of by any one who knows how to look out a logarithm in a table. They obviously depend on the form of log. 2:—

Rule I.—To find the equivalent of a given vibrations ratio in E. T. semitones. Take the common logarithm of the given ratio, subtract $\frac{1}{300}$, and call this the first improved value (F. I. V.). From the original logarithm subtract $\frac{1}{300}$ of the first improved value, and $\frac{1}{10,000}$ of the first improved value. Multiply the remainder by 40. The result is the interval, expressed in E. T. semitones.

Example 1.—To find the values of a perfect fifth, the vibrations ratio of which is $\frac{3}{2}$ in E. T. semitones:—

Log. 3	= ·4771213	·1760213
Log. 2	= ·3010300	5850
Log. $\frac{3}{2}$	= ·1760913	·1755063
$\frac{1}{300}$	= ·0005870	175
F. I. V.	= ·1755043	·1754888
		40
		7·01955|2

Thus a perfect fifth exceeds 7 semitones by ·01955.

N.B.—The rule only gives five places correct.

The true value of the perfect fifth calculated by an exact process to 20 places is:—

7·01955 00086 53874 17740

Example 2.—To find the value of a perfect third, the vibrations ratio of which is $\frac{5}{4}$, in E. T. semitones:—

Log. 5	= ·6989700	·0969100
Log. 4	= ·6020600	3219
Log. $\frac{5}{4}$	= ·0969100	·0965881
$\frac{1}{300}$	= ·0003230	96
	·0965870	·0965785
		40
		3·86314|0 = 4 − ·13686

Thus a perfect third exceeds 3 semitones by ·86314; or, as it is generally more convenient to state the result, it falls short of 4 semitones by ·13686.

The value to 20 places is:—

4 − ·13686 28613 51651 82551

Rule II.—To find the vibrations ratio of an interval given in E. T. semitones.

To the given number add $\frac{1}{300}$ and $\frac{1}{10000}$ of itself. Divide by 40. The result is the logarithm of the ratio required.

Example.—The E. T. third is 4 semitones. The vibrations ratio, found as above, is 1·25992. Hence the vibrations ratio of the E. T. third to the perfect third is nearly 126 : 125.

3. Definitions.

Regular Systems are such that all their notes can be arranged in a continuous series of equal fifths.

Regular Cyclical Systems are not only regular, but return into the same pitch after a certain number of fifths: every such system divides the octave into a certain number of equal intervals.

Error is deviation from a perfect interval.

Departure is deviation from an E. T. interval.

Intervals taken upwards are called positive, taken downwards negative.

Fifths are called positive if they have positive departures, that is, if they are greater than E T. fifths; they are called negative if they have negative departure, that is, if they are less than E. T. fifths. We saw that perfect fifths are more than 7 semitones; they are therefore positive.

Systems are said to be positive or negative according as their fifths are positive or negative.

Regular Cyclical Systems are said to be of the *r*th order, positive or negative, when the departure of 12 fifths is $\pm r$ units of the system.

Thus, we shall see later that in the system of 53 the departure of 12 fifths is 1 unit upwards, and the system is positive of the first order; in the system of 118 the departure of 12 fifths is 2 units upwards, and the system is said to be positive of the second order. In the system of 31 the departure of 12 fifths is one unit downwards, and the system is negative of the first order; and in the system of 50 the departure of 12 fifths is two units downwards, and the system is negative of the second order. Systems of the first order are called Primary, of the second order Secondary.

4. Formation of Intervals by Series of Fifths.

When successions of fifths are spoken of, it is intended that octaves be disregarded. If the result of a number of fifths is expressed in E. T. semitones, any multiples of 12 (octaves) may be subtracted or added.

As an example, we will consider some of the intervals formed by successive fifths in the system of 53. We shall see later (Theorem iv.) that the fifth of this system is $7\frac{1}{53}$; *i.e.*, it exceeds the equal temperament fifth by $\frac{1}{53}$ of an E. T. semitone. This being premised, we have the following intervals, amongst others:—

Departure of 12 fifths $= \frac{12}{53}$.

For $12 \times 7\frac{1}{53} = 84\frac{12}{53}$; and we subtract 84, which represents 7 octaves.

Two-fifths tone $= 2\frac{2}{53}$.

For $2 \times 7\frac{1}{53} = 14\frac{2}{53}$; and we subtract 12, which represents 1 octave.

* Seven-fifths semitone, formed by 7 fifths up, $= 1\frac{7}{53}$.

For $7 \times 7\frac{1}{53} = 49\frac{7}{53}$; and we subtract 48, which represents 4 octaves.

* Five-fifths semitone, formed by 5 fifths down, $= 1 - \frac{5}{53}$

For $5 \times - 7\frac{1}{53} = - 35\frac{5}{53}$; and we add 36, which represents 3 octaves.

* These expressions were suggested to the writer by Mr. Parratt.

Or, if we consider the system of 31 in which the fifth is $7 - \frac{1}{31}$, we have, similarly:

Departure of 12 fifths $= \ - \frac{12}{31}$
Two-fifths tone $= 2 - \frac{2}{31}$
Seven-fifths semitone $= 1 - \frac{7}{31}$
Five-fifths semitone $= 1\frac{5}{31}$

5. Regular Systems.

The importance of Regular Systems arises from the symmetry of the scales which they form.

Theorem i.—In any regular system 5 seven-fifths semitones and 7 five-fifths semitones make up an exact octave.

For the departures from E. T. of the 5 seven-fifths semitones are due to 35 fifths up, and the departures of the 7 five-fifths semitones are due to 35 fifths down, leaving 12 E. T. semitones, which form an exact octave.

Example.—A perfect fifth $= 7 + \delta$, where $\delta = \cdot 01955$.
Then the seven-fifths semitone is $1 + 7\,\delta$,
the five-fifths semitone is $1 - 5\,\delta$,
and 5 seven-fifths semitones, together with 7 five-fifths semitones, is:—

$$5\,(1 + 7\,\delta) + 7\,(1 - 5\,\delta) = 5 + 35\,\delta + 7 - 35\,\delta = 12.$$

Theorem ii.—In any regular system the difference between the seven-fifths semitone and the five-fifths semitone is the departure of 12 fifths, having regard to sign.

That is to say, if we subtract the five-fifths semitone from the seven-fifths semitone, the result is equal to the departure of 12 fifths in value; and it is positive if the fifths are positive, and negative if the fifths are negative.

For the seven-fifths semitone up is one E. T. semitone up and the departure of 7 fifths up, and the five-fifths semitone down is one E. T. semitone down and the departure of 5 more fifths up—which makes, on the whole, the departure of 12 fifths up, and if the single departures are positive, then the twelve departures are positive, and if negative, negative.

Example 1.—A perfect fifth $= 7 + \delta$ as before, and δ is positive.
And seven-fifths semitone $= 1 + 7\,\delta$
five-fifths semitone $= 1 - 5\,\delta$,
whence, subtracting the lower line, the difference $= 12\,\delta$.

Example 2.—A fifth of the system of 31 $= 7 - \frac{1}{31}$, and it is negative.
The seven-fifths semitone $= 1 - \frac{7}{31}$
Five-fifths semitone $= 1 + \frac{5}{31}$
whence, subtracting the lower line, the difference $= -\frac{12}{31}$.

6. Regular Cyclical Systems.

The importance of Regular Cyclical Systems arises from the infinite freedom of modulation in every direction which is possible in such systems when properly arranged; whereas in non-cyclical systems required modulations are liable to be impossible, owing to the demand arising for notes outside the material provided.

Theorem iii.—In a Regular Cyclical System of order $\pm r$, the difference between the seven-fifths semitone and five-fifths semitone is $\pm r$ units of the system.

For, recalling the definition of rth order (departure of 12 fifths $= \pm r$ units), the proposition follows from Theorem ii.

EXAMPLE 1.—In the system of 53 the fifth is $7\frac{1}{53}$;
Seven-fifths semitone $= 1\frac{7}{53}$
Five-fifths semitone $= 1 - \frac{5}{53}$,
whence subtracting, the difference is $\frac{12}{53}$, which is the octave divided by 53, or one unit of the system.

EXAMPLE 2.—In the system of 31 the fifth is $7 - \frac{1}{31}$, and, as before, the difference is $- \frac{12}{31}$, or $-$ (one unit of the system).

Corollary.—This proposition, taken with Theorem i., enables us to determine the numbers of divisions in the octave in systems of any order, by introducing the consideration that each semitone must consist of an integral number of units. The principal known systems are here enumerated:—

7-fifths semitone $= x$ units	5-fifths semitone $= y$ units	Number of Units in octave (Th. i.) $5x + 7y = n$
	Primary (1st order) Positive.	
2	1	17
3	2	29
4	3	41
5	4	53
6	5	65
	Secondary (2nd order) Positive.	
11	9	118
	Primary Negative.	
1	2	19
2	3	31
	Secondary Negative.	
3	5	50

The mode of formation in other cases is obvious.

Passing over, for the present, the derivation of scales from this scheme, we proceed to other important theorems on Cyclical Systems:—

Theorem iv.—In any Regular Cyclical System, if the octave be divided into n equal intervals, and r be the order of the system, the departure of each fifth of the system is $\frac{r}{n}$ E. T. semitones.

Let the departure of each fifth of the system be δ. Then the departure of twelve fifths $= 12\,\delta = r$ units by definition, and the unit $= \frac{12}{n}$ E. T. semitones (since the octave, which is 12 semitones, is divided into n equal parts). Hence

$$12\,\delta = r.\frac{12}{n} \text{ or } \delta = \frac{r}{n}$$

EXAMPLE 1.—In the system of 53, which is of the first order and positive (Th. iii. Cor.), the departure of 12 fifths = 1 unit, $= \frac{12}{53}$; whence the departure of one fifth $= \frac{1}{53}$.

EXAMPLE 2.—In the system of 118, which is of the second order and positive, the departure of 12 fifths is 2 units, $= 2.\frac{12}{118}$; whence the departure of one fifth is $\frac{2}{118}$, or $\frac{1}{59}$.

EXAMPLE 3.—In the system of 31, which is of the first order and negative, the departure of each fifth is $-\frac{1}{31}$.

EXAMPLE 4.—In the system of 50, which is of the second order and negative, the departure of each fifth is $-\frac{1}{25}$.

Theorem v.—If, in a system of the rth order, the octave be divided into n equal intervals, $r + 7n$ is a multiple of 12, and $\frac{r+7n}{12}$ is the number of units in the fifth of the system.

Let ϕ be the number of units in the fifth.

Then $\phi\,.\,\frac{12}{n}$ is the fifth, and $=7+\delta$, if δ be the departure of the fifth; $=7+\frac{r}{n}$ by Th. iv.

Hence $\phi=\frac{7n+r}{12}$,

and ϕ is an integer by hypothesis—whence the proposition.

From this proposition we can deduce corresponding values of n and r. This is useful in investigating systems of the higher orders. Casting out multiples of 12, where necessary, from n and r, we have the following relations between the remainders:—

Remainder of

n	1	2	3	4	5	6	7	8	9	10	11
r	5	10	3	8	1	6	11	4	9	2	7
	−7	−2	−9	−4	−11	−6	−1	−8	−3	−10	−5

EXAMPLE.—It is required to find the order of the system in which the octave is divided into 301 equal intervals.

300 is a multiple of 12; remainder 1 gives order 5, or − 7. 301 is a system of some interest regarded as a positive system of order 5, in consequence of its having, as we shall see later, tolerably good fifths and thirds; while its intervals are expressed by the first three places of the logarithms of the vibration ratios, ·3010 being the first four places of log. 2. Mr. Ellis has recently made use of this system.—(Royal Society's Proceedings, 1874.)

Theorem vi.—If a system divide the octave into n equal intervals, the total departure of all the n fifths of the system $=r$ E. T. semitones, where r is the order of the system.

For, if δ be the departure of one fifth, then, by Th. iv.,

$$\delta = \frac{r}{n}; \text{ whence } n\,\delta = r.$$

or the departure of n fifths $= r$ semitones.

EXAMPLE 1.—The departure of 53 fifths of the system of 53 is 1 semitone; for the departure of one fifth is $\frac{1}{53}$ by Th. iv.

EXAMPLE 2.—The departure of 118 fifths of the system of 118 is 2 semitones; for the departure of one-fifth is $\frac{2}{118}$.

EXAMPLE 3.—The departure of 31 fifths of the system of 31 is $-$ 1 semitone (one semitone flat); for the departure of one-fifth is $-\frac{1}{31}$.

EXAMPLE 4.—The departure of 50 fifths of the system of 50 is $-$ 2 semitones.

This theorem gives rise to a curious mode of deriving the different systems.

Suppose the notes of an E. T. series arranged, on a horizontal line, in the order of a succession of fifths, and proceeding onwards indefinitely, thus:

c g d a e b f♯ c♯ g♯ d♯ a♯ f c g . . . .

and so on.

Let a regular system of fifths start from *c*. If they are positive, then at each step the pitch rises farther from E. T. It can only return to *c* by sharpening an E. T. note.

Suppose that *b* is sharpened one E. T. semitone, so as to become *c*; then the return may be effected—

at the first *b* in 5 fifths
—— second *b* in 17 fifths
—— third *b* in 29 fifths,—

and so on. Thus we obtain the primary positive systems.

Secondary positive systems may be got by sharpening *b♭* by two semitones, and so on. If the fifths are negative, the return may be effected by depressing *c♯* a semitone in 7, 19, 31, . . fifths; we thus obtain the primary negative systems, or by depressing *d* two semitones, by which we get the secondary negative systems,—and so on.

FORMATION OF MAJOR THIRDS.

7. NEGATIVE SYSTEMS.

The departure of the perfect third is $-\cdot 13686$, as we have seen (section 2); that is to say, it falls short of the E. T. third by this fraction of an E. T. semitone. Hence negative systems, where the fifth is of the form $7-\delta$, form their thirds in accordance with the ordinary notation of music. For if, in such a system, we form a third by taking four fifths up, we have a third with negative departure $(= -4\delta)$, which can approximately represent the departure of the perfect third. Thus, *c♯* is either the third to *A*, or four fifths up from *A*, in accordance with the usage of musicians.

Example.—In the system of 31 the departure of each fifth is $-\frac{1}{31}$, and that of the third by four-fifths up is $-\frac{4}{31} = -\cdot12903$; and this differs from the departure of the perfect third ($= -\cdot13686$) only by the small error $\cdot00783$, or considerably less than the hundredth of a semitone.

8. Positive Systems.

Positive systems, on the other hand, form their approximately perfect thirds by 8 fifths down; for their fifths, being larger than E. T. fifths, depress the pitch below E. T. when tuned downwards. Thus, if the fifth be of the form $7+\delta$, 8 fifths down give the negative departure ($=-8\delta$), which can approximately represent the departure of the perfect third. Thus the third of A should be D♭, which is inconsistent with musical usage. Hence positive systems require a separate notation, to which we will return immediately.

Example 1.—Regular system of perfect fifths. The departure of a perfect fifth is $\cdot01955$, as we have seen. Eight fifths down give therefore a departure $= -8 \times \cdot01955 = -\cdot15640$; and this exceeds the departure of a perfect third ($= -\cdot13686$) by the error $\cdot01954$; a quantity which is the same, within one unit in the last place, as the departure of the perfect fifth, or the error of the E. T. fifth, which is the same thing.

Example 2.—System of 53. Departure of third by 8 fifths down $= -\frac{8}{53} = -\cdot15095$.

Example 3.—System of 118. Departure of third by 8 fifths down $= -\frac{8}{59} = -\cdot13560$, the error of which is little more than the thousandth part of a semitone.

9. Notation.

Helmholtz employs a peculiar notation for the system generally called by his name, which has very nearly perfect fifths and perfect thirds.* We shall speak of this system in general as the positive system of perfect thirds. Helmholtz's employment of this notation is marked by several peculiarities, which we need not here discuss; the objection that this notation is unsuitable for use with musical symbols is sufficient to warrant us in disregarding it.

The following notation is here adopted for positive systems in general: it is not intended to be limited to any one system, like Helmholtz's. In fact, as it consists entirely of an indication of position in a series of fifths, it may, when desired, be used with negative systems.

The notes are arranged in series, in order of successive fifths. Each series contains twelve fifths, from *f*♯ up to *b*. The series *f*♯—*b* is called the unmarked series; it contains the standard, or unmarked *c*. Each note of the next series of twelve fifths up is affected with the mark (/), which is called a mark of elevation,

* See note at p. 121, *post*.

and is drawn upwards in the direction of writing. The next series of twelve fifths up is affected with the mark (//); and the succeeding series of twelve fifths up are affected with a number of marks of elevation corresponding to their position, (///), (////), and so on. The series below the unmarked series is affected with the mark (\), which is called a mark of depression, and is drawn downwards in the direction of writing; and the succeeding series, in a descending order of fifths, are affected with a number of marks of depression corresponding to their position, (\\), (\\\), and so on. Such fifths as *\b–f♯*, *b–/f♯*, which join any two of the series of the notation, have the same value as all the rest.

Thus, for example, the interval *c–/c* is the departure of twelve fifths. *c–\e* are related through eight fifths downward from *c*. Hence in positive systems *\e* is the note which forms an approximately perfect third with *c*.

N.B.—It is to be noted that the position in the series of fifths is determined only by the notation here introduced; *i.e.*, *c♯* and *d♭* mean exactly the same thing, and refer only to one of the twelve E. T. divisions of the octave. Regarded as belonging to an assigned system, *c♯* or *d♭* would mean that note of the unmarked series which is five fifths below the unmarked or standard *c*.

10. Rule for Thirds in Positive Systems.

If we write down one of the series of the notation:—

f♯–c♯–g♯–d♯–a♯–f–c–g–d–a–e–b,

and remember that positive systems form their thirds by eight fifths down, we have the rule:—

The four accidentals on the left in any series of the notation form major thirds to the four notes on the right of the same series. All other notes have their major thirds in the next series below. Thus, *d–f♯* and *c–\e* are major thirds.

11. Employment of the Notation in Music.

This notation is suitable for employment with written music. Its appearance will be generally taken to indicate the employment of a system with perfect or approximately perfect fifths, unless anything is said to the contrary.

The following passage is an example:—

The interval *g*– *f* is a close approximation to the harmonic or natural seventh; ╱*a*♭–*f*♯ is rendered very smooth by the employment of the same interval. The development of the practical use of the notation is deferred for the present.

The notation is also useful for the discussion of some systems of historical interest. Thus, we have a scale of F in Mersenne, whose work bears the date 1636, with eighteen notes to the octave. This possessed the following resources:—

Major chords of *c*–*f*–*b*♭–*e*♭,
" ╲*e*–╲*a*–╲*d*–╲*g*, thirds to the above.
" ╱*a*♭–╱*d*♭–╱*g*♭, thirds below *c*–*f*–*b*♭.

Minor chords of *c*–*f*–*b*♭–*e*♭,
" ╲*e*–╲*a*–╲*d*–╲*g*, thirds to the above.
" ╲*c*♯–╲*f*♯–╲╲*b*, thirds to ╲*a*–╲*d*–╲*g*.

We have here the two forms of second of the key, *g* and ╲*g*, differing approximately by a comma. This double form appears in all good attempts at systems with perfect fifths.

12. The 'Generalised Keyboard.'

A keyboard has been designed and constructed, by means of which the notes of all regular systems, positive and negative, can be brought under the control of the fingers. The detailed account of this keyboard is deferred for the present. It contains eighty-four keys in every octave; the instrument of which it forms a part is a harmonium, and the system of tuning is that which divides the octave into fifty-three equal intervals. The form of fingering is the same in all keys. Such passages as the example in musical type given above can be readily performed upon it.

SYMMETRICAL ARRANGEMENT OF THE NOTES OF THOMPSON'S ENHARMONIC ORGAN.

The Subscripts 1, 2, and 3 refer to its Three Keyboards.

12	$'c_{1}$	.	.	.	.	.	.	.	.	.	.	.	$'c_{1}$
11	.	.	.	.	.	$'f_{1}$	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	$'b\flat_{1}$	.	.
9	.	.	.	$'e\flat_{1}$	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	$'a\flat_{1}$	.	.	.	.
7	.	$'c\sharp_{1}$	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	$'f\sharp_{1}$	.	.	.	.	.	.
5													
4	.	.	.	.	e_{2}	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	.	.	$a_{1\cdot2}$	.	.	.
2	.	.	$d_{1\cdot2}$	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	$g_{1\cdot2\cdot3}$	.	.	.	.	.
12	$c_{1\cdot2\cdot3}$	.	.	.	.	.	.	.	.	.	.	.	$c_{1\cdot2\cdot3}$
11	.	.	.	.	.	$f_{1\cdot2\cdot3}$	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	$b\flat_{1\cdot3}$	.	.
9	.	.	.	$e\flat_{1\cdot3}$	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	$a\flat_{3}$	.	.	.	.
7	.	$c\sharp_{2}$	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	$f\sharp_{1\cdot2}$	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.	.	.	$^{\backprime}b_{1\cdot2\cdot3}$	.
4	.	.	.	.	$^{\backprime}e_{1\cdot2\cdot3}$	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	.	.	$^{\backprime}a_{1\cdot2\cdot3}$	.	.	.
2	.	.	$^{\backprime}d_{1\cdot2\cdot3}$	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	$^{\backprime}g_{1\cdot2\cdot3}$	.	.	.	.	.
12	$^{\backprime}c_{3}$	.	.	.	.	.	.	.	.	.	.	.	$^{\backprime}c_{3}$
11	.	.	.	.	.	$^{\backprime}f_{3}$	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	$^{\backprime}b\flat_{2}$	.	.
9	.	.	.	$^{\backprime}e\flat_{2}$	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	$^{\backprime}a\flat_{2\cdot3}$	.	.	.	.
7	.	$^{\backprime}c\sharp_{2\cdot3}$	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	$^{\backprime}f\sharp_{2\cdot3}$	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.	.	.	$^{\backprime\backprime}b_{2\cdot3}$	.
4	.	.	.	.	$^{\backprime\backprime}e_{3}$	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	.	.	$^{\backprime\backprime}a_{3}$	.	.	.
2													
1	.	.	.	.	.	.	.	$^{\backprime\backprime}g_{2}$	.	.	.	.	.
12	$^{\backprime\backprime}c_{2}$	.	.	.	.	.	.	.	.	.	.	.	$^{\backprime\backprime}c_{2}$
11	.	.	.	.	.	$^{\backprime\backprime}f_{3}$	.	.	.	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	$^{\backprime\backprime}b\flat_{2}$	.	.
9	.	.	.	$^{\backprime\backprime}e\flat_{2}$	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	$^{\backprime\backprime}a\flat_{3}$	.	.	.	.
7	.	$^{\backprime\backprime}c\sharp_{3}$	.	.	.	.	.	.	.	.	.	.	.

13. Principle of Symmetrical Arrangement.

This principle is employed in the design of the above-mentioned keyboard; and it is owing to its properties that the fingering is the same in all keys. It may be thus stated:—

If we place the E. T. notes on a horizontal line in the order of the scale, and set off the departures of the notes of any system at right angles to the E. T. line, sharp departures up and flat departures down, we obtain the positions of a symmetrical arrangement. The accompanying table is a symmetrical arrangement of the notes of General Thompson's enharmonic organ. The following series of intervals lie on characteristically-placed straight lines in a symmetrical arrangement:—

Name of interval in general case.	Name when fifths are perfect.	Series of intervals.
2-fifths tone	Major tone	*c–d–e* . . .
5-fifths semitone . . .	Pythagorean semitone. .	/*c*–/*c*♯–*d*–*e*♭–*e* . .
7-fifths semitone . . .	Apotomè Pythagorica . .	*c*–*c*♯–*d*–/*e*♭ . . .
Third by 4 fifths up . .	Dissonant, or Pythagorean third	*c–e*–/*a*♭–/*c* . . .
Third by 8 fifths down .	(Approximately perfect third.)	*c*–*e*–*a*♭–*c* . .

The departure of twelve perfect fifths, or the Pythagorean comma, = 12 × ·01955 = ·23460. The ordinary comma of ($\frac{81}{80}$) is ·21506. For a certain degree of approximation we can neglect the difference between these quantities, and speak of a comma without specialising our meaning. In this sense we may read the last of the above series of intervals as showing us that three perfect thirds fall short of the octave by two commas approximately. We may return to this point hereafter.

In the symmetrical arrangement of the notes of General Thompson's organ, we may note specially, first, the effect of the distribution of the notes over these keyboards. For instance, the notes of the chord of A minor are all present ($a_{1}._{2}$–/c_{1}.–e_{2}.); but the third and fifth are on different keyboards, so that the chord would not be generally available.

Again, the notes *b* and *d* are missing from the otherwise complete scheme; we notice the number of chords which their absence destroys.

In the present paper the endeavour has been made to present to the members of the Musical Association the more fundamental portions of the theory of the subject. It is hoped that this treatment may facilitate the comprehension of such historical points and such further developments as may be hereafter brought before the Association.

MAY 3, 1875.

JOHN HULLAH, ESQ., VICE-PRESIDENT, IN THE CHAIR.

TEMPERAMENT; OR, THE DIVISION OF THE OCTAVE.

By R. H. M. BOSANQUET, M.A., F.R.A.S., F.C.S.,
Fellow of St. John's College, Oxford.

PART II.

14. Introduction.

In a previous paper read before the Musical Association on November 2, 1874, a method was developed for the derivation and treatment of a class of systems of tuning, to which the term 'regular' was applied—this term being taken to imply that the notes of any such system can be arranged in a continuous series of equal fifths. A notation was described, applicable to written music, by which the position of the notes of such systems in the fundamental series of fifths is defined; and a brief sketch was given of a 'Generalised Keyboard,' founded on a principle of 'symmetrical arrangement,' by means of which the notes of any such system can be controlled, the fingering of any passage being the same in whatever key it is taken.

In the present paper some points of interest in the history of the subject will be first alluded to. The history even of so obscure a subject is very extensive.

In Mr. A. J. Ellis's papers on 'Musical Chords,' and 'On the Temperament of Instruments with Fixed Tones,' in the Proceedings of the Royal Society for 1864, there is a concise account, with references, of the greater part of what is attainable in the history of the derivation of the octave. The points to be now mentioned will be selected with a view to illustrate the development and progress of the subject, and to supplement the information obtainable from Mr. Ellis's papers.

In the discussion of Helmholtz's work, his theory of consonance and dissonance will be examined in some detail, and incidentally the simplest method of computing beats for practical purposes will be introduced. We shall then proceed with the practical application of our systems, and the construction of instruments for their control.

15. Historical.

The practical interest of what remains to us of the theory of the Greek musicians is but small; and it will be convenient to pass it by with the remark that, rightly or wrongly, series of perfect fifths, and the principal derivatives of such a series, bear the name of Pythagoras; the Pythagorean comma being the difference in interval between the extremities of a series of twelve perfect fifths; the Pythagorean, or dissonant third, the third made by four perfect fifths up, and other intervals formed by the notes of such a series bearing analogous names. The approximately true third, however, formed by eight fifths down, has never borne the name of Pythagoras.

16. We pass at once, with this observation, to the beginning of the seventeenth century. The work of Mersenne, bearing the date 1636, affords us information as to the state of the problem at that time.[1]

The portion of Mersenne's work with which we are most directly concerned treats of systems of scales. Almost the whole of these are constructed so as to afford a greater or less number of perfect concords. We select the scale of the key of F, with 18 intervals in the octave. Mersenne dignifies it, and many other of his systems, with the title of 'Systema Perfectum.' We employ the signs in his figure of the keyboard at p. 118. The keyboard is identified as belonging to the scale by the proportional vibration numbers, which are given for each note in both places.

The following is Mersenne's table :—

Systema Perfectum.		Positions according to System of Perfect or Approximately Perfect Fifths.
F	5760	*f*
	Semit. majus	
E	5400	*e*
	Semit. minus	
·× E	5184	/*e*♭ or /*d*♯
	Comma	
× D	5120	*e*♭ or *d*♯
	Semit. majus	
D	4800	*d*
	Semit. minus	
× × D	4608	/*d*♭ or /*c*♯
	Diesis	
× C	4500	*d*♭ or *c*♯
	Semit. minus	
C	4320	*c*
	Semit. majus	
♮	4050	*b*
	Comma	
× ♭ or ·B	4000	*b*
	Semit. minus	
B	3840	*b*♭
	Semit. majus	
A	3600	*a*
	Semit. minus	
× × A	3456	/*a*♭ or /*g*♯
	Diesis	
× G	3375	*a*♭ or *g*♯
	Semit. minus	
·G	3240	*g*
	Comma	
G	3200	*g*
	Semit. minus	
× × G	3072	/*g*♭ or /*f*♯
	Diesis	
× F	3000	*g*♭ or *f*♯
	Semit. minus	
F	2880	*f*

[1] The works of Salinas and Zarlino, in the 16th century, are not accessible to the writer. Salinas is said to have invented, and Zarlino to have first published, the mean-tone system.

For the exhibition of the resources of this system, as regards the chords available, we must refer to the first part of this paper (p. 15). We see how limited these resources are, and yet how judiciously the most is made of the limited number of notes provided.

The keyboard for this scale is figured at p. 118 of the book 'De Instrumentis,' in Mersenne.

All the black keys are doubled, and also the key for the G. Two pairs of black keys are placed one pair on each side of the G key, and three pairs between *b*—*c*, *c*—*d*, and *d*—*e*, respectively, instead of in the usual positions.

Mersenne commences his systems with some examples taken from Salinas. The first system is one of 25 notes in the octave, which is given in a form precisely analogous to the above; this he calls 'Systema Omnium Perfectissimum.'

This and all other similar systems are on the principle of the addition of more notes, which furnish perfect concords with some of those already present. The best-known and most-developed example of this kind in recent times is the enharmonic organ of the late General T. Perronet Thompson. The resources of such a system are augmented by the addition of every new note in exactly the same manner. General Thompson employed altogether 40 in the octave. It is to be noticed that these are all irregular systems; there is no pretence of aiming at a continuous series of fifths. The fifths, being perfect, are of the character of those of positive systems.

Mersenne alludes to negative systems, although none such are fully explained by him. He mentions the system produced by dividing the octave into 31 equal intervals. He states that it is obtained by dividing the whole tone into five equal parts. Now, six major tones exceed an octave by the Pythagorean comma. Hence Mersenne's statement is not true. But, if we diminish the value of the major tone in a certain ratio, we can make six tones fall short of the octave by a fifth part of a tone; and then we have $6\times5+1=31$ fifths of a tone in the octave. Although the existence of the system was known to Mersenne, and indeed also to Salinas, they both mention it only to state that they regard it as defective, and it was left to Huyghens to recognise its importance. Mersenne then mentions the equal temperament of 12 notes very shortly, and also a division of the octave into 24 equal intervals. The latter is useless, and he does not appear to have recognised any merit in the former, save its simplicity.

At pages 66, 118, 119, and 129 are given representations of keyboards for systems having various numbers of notes in the octave. These are interesting, and some of them are of very great complexity.

On page 128 there is a very remarkable table, in which the rules for the dimensions of organ-pipes are set off for a number of different systems. The equal temperament positions are given; and in the third column the positions are given for a system, which is not further explained, but appears to be substantially the old unequal temperament. Although this system was certainly known at the time, it was not apparently as yet in favour with theorists.

17. The next writer worthy of attention is Huyghens. In a tract called 'Cyclus Harmonicus' (Opera Varia, Vol. I.) he treats of the above-mentioned system of 31. He notices that Salinas and Mersenne had not the knowledge of the methods necessary for its construction, and gives a correct solution by the aid of logarithms. The work appears to have been first published before 1700, but the date of the Opera Varia is 1724.

18. An important work in the history of the subject is 'Smith's Harmonics.' The date of the second edition is 1759. Among the remarks in the preface, a curious one deserves attention, especially as illustrating the mode of thought of practical men at that time. It consists in tempering the major third according to the rule :—

$$\frac{\text{Interval of octave}}{\text{Interval of major third}} = \text{ratio of circumference of circle to diameter.}$$

If we seek to test this we find the proportion :—

$$\frac{12 \text{ semitones}}{3{\cdot}14159} = 3{\cdot}81972 \text{ semitones,}$$

$$\text{and perfect third} = \underline{3{\cdot}86314}\,;$$

whence the third thus derived is $\cdot 04342$ flat.

The properties of the intervals have of course nothing to do with the circle, and the approximate numerical coincidence is merely accidental.

In speaking of the system of mean tones, Smith observes that its characteristic is that all the fifths are one-fourth of a comma flat; for thus the third formed by four fifths is a comma below the Pythagorean third, which is the property of the perfect third. He calls it 'the vulgar temperament'; it is the old form of the unequal temperament. The object at which he aims in the systems he then proposes is, to get the thirds, fifths, and sixths equally tempered, so that the thirds and sixths may beat as fast as the fifths. From our point of view this is of doubtful correctness. We find that fifths are much more sensitive to temperament than thirds, whereas Smith inclines to the opposite view, but eventually adopts the principle that all concords shall be made equally dissonant. This he calls 'equal harmony.'

With reference to the remark made just now, that fifths are more sensitive to temperament than thirds, we must note that the opposite opinion is sometimes expressed. Let us, therefore, consider shortly how this stands. In the ordinary equal temperament the thirds are sharp by about two-thirds of a comma, and we must admit that this amount of error in the thirds is not readily detected unless the ear be specially attracted to it; for all practical musicians use these intervals constantly without perceiving the dissonance. It is no doubt matter of opinion to some extent, but the writer's own experience is that fifths which are as much out of tune as two-thirds of a comma are unbearable. The beats in the two cases differ in two ways. Those of the fifth are far more intense than those of the third, since they are derived from an interfering pair of harmonics which lie lower in pitch—namely, the twelfth and octave, and in musical tones these are always more intense than higher harmonics; but these beats are less rapid than those of the third, which are derived from the interference of tierce and double octave. No doubt, therefore, the result depends upon the strength of the various harmonics present, and may vary in different cases.

Smith does not appear to regard positive systems as practicable; on the ordinary keyboard this is undoubtedly true.

Smith gives a general investigation of the properties of negative systems of temperament. His method is interesting, as it furnishes immediately a result which we shall obtain in a different manner. He assumes that the octave is made up of five similar tones and two semitones. We see that this excludes positive systems, with their two classes of tones. He then points out that, since the five tones and two semitones make up an octave, which is a constant quantity, if we suppose the values of the tone and semitone to be changed in any manner, the change of the five tones must be equal and opposite to the change of the two semitones, or the change of the tone is to that of the semitone as 2 : 5. Now in treating negative systems we shall see that all the tones are what we call two-fifths tones, *i.e.* they are got by tuning two fifths up; and both the semitones are five-fifths semitones, *i.e.* are got by tuning five fifths down. Hence the departure from equal temperament of the tone is that of two-fifths up, and of the semitone five fifths down—whence Smith's result follows directly. He examines the systems of 50 and 31 by this method.

Smith gives a very detailed investigation of the theory of beats. This was the most important discussion of the subject up to recent times, and De Morgan commends it highly. The investigations of Helmholtz, however, with the recognition of Ohm's law of simple tones, have disposed of the fundamental principles he employs so far as regards the reception of sound in the ear; and consequently Smith's discussion becomes unimportant for us.

Smith gives a number of rules for determining absolute pitch, or the number of vibrations made in a given time by any note. The first of these depends on the mechanical properties of the monochord, the rest on the observation of beats. None of them are of practical value now, as they do not admit of being performed with sufficient accuracy for our purposes.

As Smith's systems led to a very unequal temperament, they are valueless, practically, without additional notes to the keyboard. His direction for ordinary keyboards is as follows:—

'Till instruments are made with a changeable scale, it is more proper to tune the defective scale in common use by making every fifth and third to the same base beat equally quick—the former flat, and the latter sharp.'

This system is intermediate between the old unequal temperament, or system of mean tones, and the ordinary equal temperament. It is an improvement on the former. The more recent form of the unequal temperament is roughly Smith's system. (See Hopkins on the Organ, p. 183.) Smith describes (p. 177) an arrangement for introducing additional notes into the harpsichord, to be substituted for the normal notes on certain keys by the action of drawstops. There is an organ in a private house in Edinburgh which possesses a similar arrangement. In both cases the notes are intended to constitute a regular negative system, either the old unequal temperament (mean-tone system) or an approximation to it. The action of the stops was to vary the portion of the circle of fifths presented on the keyboard. Thus the following arrangements might be accessible with two drawstops, the one extending the resources of flat keys, the other those of sharp keys:

Flats: *b–e–a–d–g–c–f–b♭–e♭–a♭–d♭–g♭*

Ordinary arrangement: *g♯–c♯–f♯–b–e–a–d–g–c–f–b♭–e♭*

Sharps: *e♯–a♯–d♯–g♯–c♯–f♯–b–e–a–d–g–c*

Mr. A. J. Ellis has recently caused to be constructed a harmonium in which this principle is further developed. (See Proceedings of the Royal Society, 1874.) The writer's opinion is, that any procedure which involves the use of mechanism, whether stop-handles or pedals, for the selection of the notes required, is unpractical.

There is an interesting passage in which Smith anticipated to some extent the doctrine of Helmholtz, that dissonance consists of beats, or intermissions more or less rapid in the continuity of the musical tone. He says, at p. 227, using expressions almost identical with those of Helmholtz:—'For nothing gives greater offence to the hearer, though ignorant of the cause of it, than those rapid rattling beats of high and loud sounds, which make imperfect consonances with one another; and yet a few slow beats, like the slow undulations of a close shake now and then introduced, are far from being disagreeable.'

19. An important tract is Woolhouse's 'Essay on Musical Intervals' (1835). This writer employs, in the first instance, equal temperament semitones, which he calls mean semitones, as his unit of interval; but subsequently abandons the method for another, which we now judge to be of less utility. Woolhouse gives numerical values of the principal intervals in E. T. semitones to twelve places; these agree, as far as they go, with the twenty-place values given in the first part of this paper.

The solution of the general problem given by Woolhouse is chiefly interesting on account of the principles employed in the actual computation. In the result, he divides the octave into fifty equal intervals, and employs only a certain number of these to construct scales. He provides a considerable number of scales, but, in consequence of the omission of many notes of the system, these do not fulfil the important conditions of a regular system, still less those of a regular cyclical system. The distinction between the sharp of a note and the flat of the note above it is retained and made essential. We have seen that this form of notation can only be employed in negative systems.

Woolhouse then (p. 49) treats the Huyghenian division of 31 in the same way, and finishes by dividing the octave into 19 equal intervals. The resulting concords would be bad, and the sequences of melody also. This system is negative, and of the first order.

Woolhouse's explanation of the 'terzo suono,' sometimes called Tartini's tone, is erroneous, as are all explanations before that of Helmholtz; otherwise, as a treatise on musical mathematics, this tract is of great excellence; it deserves to be read still as a model of elementary exposition.

20. A system was proposed by Earl Stanhope, of which the scale is substantially as follows: *c*–*g*–*d*–╲*a*–╲*b*–*f*♯–╱*g*♭–╱*d*♭–╱*a*♭–╱*e*♭–*b*♭–*f*–*c*. Its capabilities are more limited than those of any of the scales of Mersenne.

21. A paper by De Morgan, 'On the Beats of Imperfect Consonances' (Cam. Phil. Trans. Vol. X. p. 129), contains some reference to our subject. He employs equal temperament semitones as the unit of interval, and gives rules for the transformation of logarithms of vibration ratios into terms of equal temperament semitones and *vice versâ*, identical in principle with those in the writer's previous paper.

The rest of De Morgan's paper is foreign to the present subject; but we may remark that his treatment of the problem of beats belongs to the period which preceded Helmholtz's investigations.

22. A paper by Herschel (Quarterly Journal of Science, Vol. V., p. 338) proposes various scales. In one of these, which is preferred, all the fifths except one are perfect, the remaining one erring of course by a Pythagorean comma. His observation is, 'The chief blemish is the paucity of perfect thirds of both kinds, but, on the other hand, none of them err in excess or defect beyond a comma.'

The defective fifth is taken to be *d*—╲*a* (table at top of p. 348, column D, and row Sol). Now an error of a comma in a fifth makes it unfit for use in music; no ear can tolerate a fifth which is a comma out of tune. So that this arrangement would exclude from use the keys of G major, D major, and A major, to say nothing of minor keys.

23. A considerable work has been carried out by Mr. H. W. Poole: his last paper is in 'Silliman's American Journal,' Vol. XLIV., July 1867, p. 1. It is difficult to convey a good idea of the details of the keyboard he there proposes. But the general principle is as follows:—

There are five distinct series of notes, each proceeding by successive fifths. These are:—(1) keynotes, (2) thirds, (3) sevenths, (4) thirds to (2) for the minor keys, (5) sevenths to (2) for the minor keys.

The keynotes in each key are those which form perfect fifths derived from the tonic. Thus, in the key of C, we have *f–c–g –d*. The arrangement of the notes of this series is fundamental, and all the others are placed with reference to them.

These so-called keynotes are, in fact, placed according to the writer's principle of 'symmetrical arrangement.' This is, however, applied only to the 'keynotes,' and the remaining notes of each scale are introduced as auxiliaries. The relative displacements of the keynotes from the horizontal are much greater than in the writer's keyboard—three times as large. It may be as well to mention that the writer's invention of this arrangement was entirely independent of Mr. Poole's.

As far as can be judged from the description, the uniform scale of Mr. Poole's keyboard must be rather like that of A major on the ordinary keyboard. But the number of auxiliaries is considerable, and must lead to constant modification of the principal form of scale.

The writer judges the distinction of the five series to be unnecessary and cumbrous, as there exist regular systems which afford approximations sufficiently close to make one series of fifths serve for all purposes. Moreover, even the scale of 19 notes of Mersenne provides notes omitted in this arrangement—viz. the notes to which the keynotes are thirds. Mr. Poole has indeed indicated where they may be placed as auxiliaries, but they cannot be used as keynotes. The chief objection to this form of keyboard consists in the difficulty which arises from the difference in the arrangement of the auxiliary notes and the so-called keynotes. The player may want to make any note on the instrument a keynote, or to take it in any other form at a moment's notice; and this can only be accomplished perfectly by a keyboard which is symmetrical in the arrangement of all the notes it governs.

24. General T. Perronet Thompson constructed an enharmonic organ in which a number of notes belonging to the positive system of perfect thirds were commanded by three keyboards of great complexity. On each of these keyboards a scale of the ordinary keyboard is the basis of operations. A certain number of auxiliary keys are then introduced on each board, for the performance of keys related to the principal key of the board. These are introduced in a number of different forms, wherever they are wanted. The result is a complication of scales for a considerable number of different keys, compared to which the twelve scales of the ordinary keyboard are simplicity itself.

A symmetrical arrangement of the notes of this instrument was given at p. 16 of the first part of this paper.

Notwithstanding the complexity of the keyboards, performers appear to have been tolerably successful with this instrument.

In General Thompson's tract on 'Just Intonation,' the principal practical point treated of is the double second of the key. It is easy to see that the second which forms a perfect fifth to the dominant (as D in the key of C) cannot be a perfect fifth to the sixth of the key (A, determined as a perfect third to F); but we require another note, D, a comma flatter than D, to perform this function.

General Thompson employed the notation of the system of 53 in his calculations. He appears to have been aware that his system was really different, and he employed the notation of the system of 53 always within such limits as prevented the occurrence of any error in practice.

General Thompson's account of the way in which he employed the monochord is well worth attention; to vary the pitch he varied the weight by which the string was stretched, as well as the length of the string. His form of the instrument is probably the most perfect ever constructed.

25. We now come to Helmholtz. While there is in his writings but little that is strictly new on the division of the octave, he has yet so entirely revolutionised the theory of consonance and dissonance, that the modern point of view of the whole subject probably owes more to him than to any other single individual.

The theorem of the approximate identity of the major third given by eight downward steps in a perfect series of fifths and the perfect major third is given by Helmholtz; he also considers the plan of making the third perfect and of supposing the minute error that results to be distributed uniformly among the fifths, thus forming the positive system of perfect thirds. He ascribes the first knowledge of this approximate representation of the third to the Persians, on the authority of a division of the monochord recorded in one of their ancient musical treatises.

Helmholtz constructed a harmonium in which twenty-four tones to the octave were distributed between two ordinary manuals. The tones were intended to form a portion of a regular positive system of perfect thirds, but according to the rules given for the tuning the result would not be quite regular. The instrument was available for experiment, but to a very small extent only for practical purposes.

In the Seventeenth Appendix (ed. 3) there is given a plan for arranging a system of the same kind on one manual, with change-pedals or other accessory mechanism, by which the manual could be put in tune in any key that is desired The writer, however, judges this and all similar systems to be essentially defective, especially when applied to positive systems.

Helmholtz's notation [1] is not identical with that adopted in this paper. The necessity for change arose in the first instance from the impossibility of using Helmholtz's notation in written music.

[1] This notation refers primarily to perfect fifths and thirds; its development is due to Von Oettingen. Page 438, 3rd edit.

But Helmholtz's great work is the physical theory of consonance and dissonance established by him, founded on his theory of the reception of musical sounds in the ear. It is necessary here to enter into the subject to some extent to introduce the simplified forms of computation of beats which follow, and more especially as there appears to have been some oversight in the exposition of an important portion of Helmholtz's theory of hearing in the most important recent popular work on the subject.

Every musical note may be regarded as the sum of a series of simple tones, consisting of pendulum oscillations, whose vibration numbers are multiples of that of the fundamental.

When any complex system of simple tones falls on a continuous resonant scale, such as is approximately furnished by a harp, each simple tone seeks out and impresses itself upon that portion of the scale which agrees with it in pitch, and affects also the regions immediately adjoining to a greater or less extent; but beyond a certain small region surrounding the point of corresponding pitch, a simple tone does not affect the resonant scale.

In the ear a resonant scale of this description is believed to exist, and on this theory simple tones of different pitch impress themselves on different portions of the nervous organisation of the ear.

No sensible beats can be produced in the ear by interference of fundamentals separated by intervals greater than a minor third. (This limiting interval varies slightly towards both the extreme portions of the scale.) This point cannot be too strongly insisted upon. It is the basis of Helmholtz's theory; and we must digress for a moment to comment on an oversight in an exposition of the subject by a high authority.

26. In 'Tyndall on Sound' (2nd ed. p. 296) we find the following statement as the reason why no beats are audible in the octave C_1—C_2 :—

'Here our rates of vibration are 512—256; difference=256.'

'It is plain that in this case we can have no beats, the difference being too high to admit of them.'

The reason, then, is here said to be that the beats are more rapid than 132 per second, which is supposed to be the limit of them.

This is, however, not Helmholtz's position. Helmholtz's theory is that notes an octave apart affect different portions of the nervous mechanism of the ear, and consequently no beats ever take place between these sounds at all when they are received in the ear.

Attention was directed to this passage by a recent paper of Professor Mayer, of New York ('American Journal of Science,' October 1874). He does not clearly point out that the above passage departs from Helmholtz's theory, but he sees that it is wrong, and accounts for it by a new and most valuable portion of the theory, which he for the first time presents in the above paper.

The following passage from Mayer's paper puts the point clearly:—

'But if Professor Tyndall had taken, in place of the above forks, two forks giving 40 and 80 vibrations per second, he would, according to his premisses, have made this octave a most dissonant interval; for would he not have had ($80-40=40$) forty beats per second entering his ear? Similarly, if we assume that 33 beats per second always produce the maximum dissonance, then even the interval $C_1 : C_2$'—in our notation, $C-c$

—'which gives a difference of 64, is far removed from consonance.'

Mayer accounts for the difficulty by his experiments on the limits of rapidity of audible beats in different parts of the scale, which give interesting results, and complete by experimental evidence a portion of the theory, the nature of which Helmholtz had indicated only in a general manner.[1]

We cannot here enter into the details of Dr. Mayer's paper; but we must note that the numbers given furnish an actual demonstration of Helmholtz's hypothesis of the analysis of tones in the ear. Again, Helmholtz does not say, as Tyndall makes him, that beats blend always into a continuous sound when they attain the limit of 132 per second. He says (3rd ed. p. 270) of the 132 beats per second, produced by the interval b_3 c_4, 'and these are really audible, in the same manner as the 33 beats of b_1 c_2, although they sound somewhat weaker in the higher position.'

For Helmholtz's exposition of his theory we may refer to p. 271. He there points out, that if the offensiveness of beats depended on their number alone, the semitone b_1 c_2 would be as dissonant as the fifth C G, both intervals giving in air 33 beats per second; and he proceeds (p. 272) to account for this by the assumption of a law for the sympathy of the organ of Corti, according to which the intensity of the interference of fundamentals is about $\frac{1}{6}$ of the maximum for the minor third, $\frac{1}{10}$ for the major third, and insensible for all greater intervals.

It is to be noted that in the third edition Helmholtz abandons the theory that the organ of Corti is the mechanism of reception, and transfers this to the membrana basilaris, the theory of which is given in Appendix XI. In a paragraph on p. 230, Helmholtz points out that he has not altered his phraseology throughout the book, but desires it to be understood that he considers the resonance of the organ of Corti to be only such as it receives from its connection with the membrana basilaris. To return to our sketch of Helmholtz's principles.

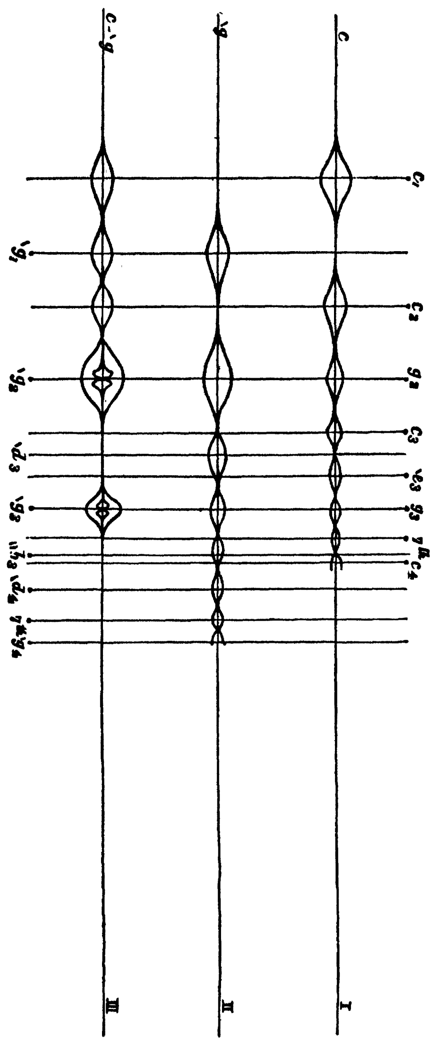
27. The beats of imperfectly-tuned consonances greater than the minor third are, in general, produced either—

(1) By the interference of harmonics present in the two notes, in pairs nearly corresponding in pitch. Or—

[1] Compare 'Philosophical Magazine,' Vol. XLIX., p. 352.

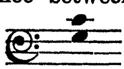
(2) By the interference of difference tones (Tartini tones) with each other, or with harmonics, in pairs nearly corresponding in pitch.

The following figure illustrates the mode of production of the beats of a fifth about a comma out of tune, by interference of harmonics, difference tones being disregarded :—



The horizontal lines are supposed to be made up of resonant points, vibrating in vertical lines. We may suppose the horizontal line to be a section of a sheet of wires standing at right angles to the paper, and somewhat similar to the sheet of wires which exists in the pianoforte.

The first figure shows the way in which a musical note is decomposed into simple tones of the harmonic series when received on an instrument of this kind. The second figure is similar, but placed about a fifth higher; while the third figure represents the combination of the two, it being supposed that the fifth is about a comma flat. The loops on the horizontal lines indicate the limits between which the points on that line may be supposed to vibrate.

The bearing of this theory on the deviation of elementary concords is obvious, but it is impossible to dwell on it here. We must only say that it differs entirely from the mode of derivation from harmonic series initiated by Dr. Day, both in that it refers the question of consonance and dissonance to a physical cause, viz., the production of beats, and also in that it takes account of the series of harmonics which exists in each note of the combination. We may refer for one moment to the explanation of the interval of the fourth as tuned free from beats, which is a crucial instance of the difference between the two points of view. If the fourth be g_0—c_1  the double octave of g_0 and the twelfth of c_1 will both fall on g_2; and if the two notes are not exactly in tune, beats will arise on that note exactly as shown in the above figure. So long as, according to Dr. Day's view, the harmonics of the lower note are alone regarded, it is impossible to assign any physical cause for the beats which arise when this interval is imperfect, as also to refer the fourth to any analogy or rule whatever; as there is no such interval as the fourth in the harmonic series, and the mere name, inversion of the fifth, conveys no explanation.

The problem is thus reduced to the interference of two sounds near each other in pitch. The beats of the lowest pair of harmonics, which approximately coincide, are always so much the strongest that we may neglect the others.

In applying these principles to the case of any imperfect concord, we notice that, if two notes are sounded together such that the vibrations numbers of the fundamentals are nearly in the ratio $p : q$, then the lowest pair of interfering harmonies will be the harmonic of the qth order in the first note, and the harmonic of the pth order in the second. It must be understood that the *order* of a harmonic is denoted by its vibrations ratio to the fundamental.

Thus in the above case of an imperfect fifth, where the fundamentals are nearly as 2 : 3, the twelfth, or harmonic of the third order of the first note, interferes with the octave, or harmonic of the second order, of the second; and in the case of the fourth

(3 : 4), the double octave, or harmonic of the fourth order, of the first note interferes with the twelfth, or harmonic of the third order, of the second. This is obvious when realised, but it often saves trouble to note the rule. The number of beats given by two interfering notes is, of course, the difference of their vibration numbers.

As an example, let us determine the number of beats per second in the equal temperament fifth c_1,—g_1, (c_1=256.) The perfect fifth being 7·01955000 semitones, the g_1, and therefore, also, its octave g_2, is ·0195500 flat. Applying rule 2, paragraph 2, we find the logarithm of the vibrations ratio of this interval:—

$$
\begin{array}{rl}
& \cdot 0195500 \\
& 0000652 = \frac{1}{300} \\
& 19 = \frac{1}{10000} \\
\hline
40) & \cdot 0196171 \\
\hline
& \cdot 0004904 = \text{log. ratio required (flattening of tempered fifth).}
\end{array}
$$

c_1 has 256, and g_2 has three times as many vibrations.

$$
\begin{array}{rll}
& 2\cdot 8853613 = \text{log. } 768\cdot 000\ (g_2) \\
\text{Log. ratio} & \cdot 0004904 \\
\hline
& 2\cdot 8848709 = \text{log. } 767\cdot 133\ (\text{tempered } g_2) \\
& \text{diff.} = \quad \cdot 867 \text{ number of beats per second.} \\
& \qquad 60 \\
\hline
& \qquad 52\cdot 04 \text{ number of beats per minute.}
\end{array}
$$

For the theory of difference of tones we must refer to Helmholtz. We shall here accept the law that the vibrations number of the difference tone is the difference of those of its primaries.

Example.—To find, the number of beats per second of the difference tones of c_1-e_1, e_1-g_1, in the equal temperament triad.

From the above example we have $c_1 = 256$, $g_1 = \frac{767\cdot 133}{2}$ $= 383\cdot 566$. We have for the ratio $c-e$ (paragraph 2)

$$
\begin{array}{rl}
\text{log. ratio} & = \cdot 1003433 \\
\text{and log. } 256 & = 2\cdot 4082400 \\
\hline
& 2\cdot 5085833 = \text{log. } 322\cdot 540
\end{array}
$$

$$
\begin{array}{lll}
& 322\cdot 540 & \\
& 256 & \\
\hline
\text{difference } e-c & 66\cdot 540 & 383.566 \\
& 61\cdot 026 & 322\ 540 \\
\hline
\text{beats per second} & 5\cdot 514 & 61.026 \text{ difference } g-e.
\end{array}
$$

Hence these difference tones give about $5\frac{1}{2}$ beats per second. In cases where the difference tones are strong, these beats, or the displacements of the difference tones, where beats are not produced, are the most disagreeable effects of the equal temperament. If the chord were in perfect tune the two difference tones would coincide, both having the vibrations number 64.

Distinction between beating dissonances and unsatisfied combinations.

We have so far assumed, with Helmholtz, that dissonance is entirely due to the presence of audible beats. Now, consider two notes separated by the interval of the harmonic seventh; these may be represented very nearly by $c - \diagdown b\flat$. When the interval is exact, beats cease entirely, and we cannot say that any dissonance, in the ordinary sense of the term, is present. We might call such combinations as this '*unsatisfied combinations*,' as distinguished from dissonances. The fourth may also be regarded as an unsatisfied combination. The weight of authority in technical music is, however, so greatly in favour of regarding such combinations as dissonances, that it appears to be necessary to comprise under the general head of dissonances both beating dissonances and unsatisfied combinations.

28. There remains to be noticed, last but not least, the series of papers by Mr. Alexander Ellis, F.R.S., in the Proceedings of the Royal Society for 1864, to which allusion has been already made, and especially the paper at p. 404, 'On the Temperament of Musical Instruments with Fixed Tones.' This paper contains a mass of valuable information on our subject. The various systems are treated according to a system of formulæ, with the object of comparing their excellence in various respects, and the results are tabulated. One point is especially worthy of attention—viz., the treatment of variations in the melodic sequences of the different systems. It is assumed (*l.c.*) that the sequences of the diatonic scale are the best, and that in proportion as the sequences of any other scale differ from those of the diatonic scale, they will offend the ear. Now this is a most important point in the theory of temperament, and one which has not received sufficient attention in general.

In support of Mr. Ellis's view, we might refer to an experiment conducted by Helmholtz and Herr Joachim, with the assistance of Helmholtz's harmonium with pure scales, which was mentioned above. The result appeared to prove conclusively that the eminent violinist employed pure scales. It may be inferred from this that the sequences of the diatonic scale would commend themselves naturally to the ear; on the other hand, this may be explained by the fact that some violinists learn their stopping by practising thirds; of course, if they play so as to get the thirds and fifths perfect, they would play substantially diatonic scales.

The writer believes, on the contrary, that the preference of certain melodic sequences by the ear is entirely a matter of education and custom. There can be no doubt that nations exist, whose tunes contain sequences which may be called quarter-tones, entirely foreign to all our ideas of music. The Egyptian Arabs are mentioned as an instance. The writer's experience with the enharmonic harmonium, on which all manner of scales producible from regular systems with practically perfect fifths can be performed, leads to the belief that, generally, to highly educated ears, the uneven sequences of the diatonic scale are disagreeable. This may be attributed to the adoption of the tempered scale as a standard, through custom. Those notes which deviate widely from the tempered scale—as, for instance, the approximate harmonic sevenths—however smooth the chords, are always found disagreeable by listeners of this class. The true minor third in the minor scale is often objected to also, as too high in pitch; it deviates very widely from the tempered note. The writer is therefore disposed, for the present, at all events, to accept the equal temperament as a better standard for melodic sequences than the diatonic scale.[1] Experimentally, indeed, it appears that the sequences of the Pythagorean scale, with dissonant thirds, are pleasanter than either, so long as harmony is kept out of the way. As the result of this discussion, we banish all consideration of melodic sequences from our theory, as an element of preference between systems, believing that the appreciation of any sequences whatever is possible with custom. As the result of his whole discussion, Mr. Ellis gives the preference to the mean-tone system, which is substantially a regular extension of the old unequal temperament, and he proposes a keyboard for its control, derived from the ordinary keyboard by the introduction of additional notes.

Another discussion of the properties of certain systems by Mr. Ellis will be found in the Proceedings of the Royal Society for 1874.

[1] The opinion above expressed, that equal temperament is the best melodic standard at present, owing to the education of musicians in that system, received remarkable confirmation at the reading of this paper. As will be mentioned later, both the just and mean-tone systems were illustrated by actual performance; and amongst numerous remarks of a similar kind, the following is perhaps the best illustration. The 33rd prelude of Bach's 48 being performed in the mean-tone system, the semitone $d\sharp - e$ was objected to by eminent musicians present. Now this semitone is, theoretically, 1·171, or very nearly $1\frac{1}{6}$ of an E.T. semitone. The just semitone (difference between perfect third and fourth) is 1,117, or between $1\frac{1}{8}$ and $1\frac{1}{9}$. Hence the mean-tone semitone is nearer to the just semitone than is the E.T. semitone. If, therefore, the diatonic scale were the true standard of melody, the mean-tone semitone should be a little better than the E.T., which would be contrary to the above observation. Again, as to the effect of education. There can be no doubt that in Handel's time all organs in England, and most elsewhere, were tuned to the mean-tone system. Bach abolished this system in Germany, and it has nearly disappeared in England. But there is no doubt that the scales of that system, which afforded good chords, were always looked upon as the best attainable in any manner, the 'wolf' due to the limited number of notes being the sole reason for discarding the system. Consequently, the semitone now objected to must formerly have been very generally received as the correct semitone. If musical education in theory were so conducted that the sounds of harmonics, intervals, &c. were studied, as well as numbers and names which have no musical value when taken alone, it might be expected that the practical objection to these intervals, which arises from the exclusive study of equal temperament, would disappear.

29. Scales.

We have seen that the intervals of regular systems are to be regarded as formed by successions of fifths. We have, therefore, only to determine the departure of the fifths of any system from equal temperament, and the number of fifths by which we proceed from the keynote to any given note of the scale; we then know at once the interval which any such note makes with the keynote.

30. Positive Systems.

Second of the Key.—In any positive system the second of the key may be derived in two ways: first, as a fifth to the dominant, in which case the derivation is by two fifths up from the keynote; and, secondly, as a major sixth to the subdominant, in which case the derivation is by ten fifths down from the keynote. Thus, the first second to *c* is *d*; the other, $\searrow d$. On account of the importance of this double form of second, we will consider the derivation of these two forms by means of the ordinary ratios, in the case, namely, in which perfect intervals are employed.

First, two fifths up and an octave down give $\left(\frac{3}{2}\right)^2 \times \frac{1}{2} = \frac{9}{8}$, when the fifths are perfect.

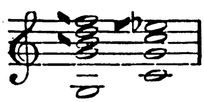
Secondly, one fifth down gives the subdominant (*c*–*f*), and a sixth up gives the depressed second ($\searrow d$), or $\frac{2}{3} \times \frac{5}{3} = \frac{10}{9}$, which is the ratio of $\searrow d$ to the keynote, when the fifths and thirds are perfect.

The ratio of $d : \searrow d$ is then $\frac{9}{8} \div \frac{10}{9} = \frac{81}{80}$, which is an ordinary comma.

We must remember that our systems only give approximations to this result, but the best of these approximations are very close.

In the harmonium, with the system of 53—which may be regarded for practical purposes as having perfect fifths, and very nearly perfect thirds—the exchange of *d* for $\searrow d$ in the chord *f*–$\searrow a$–$\searrow d$, or even in the bare sixth, *f*–$\searrow d$, produces an effect of dissonance intolerable to any ear.

Minor Third.—The minor third is not an interval which is very strictly defined by beats. In chords formed of successions of minor thirds, almost any form of the interval may be employed; and, as a matter of fact, the minor third which comes below the harmonic seventh in the series of harmonics (7 : 6), is one of the smoothest forms of this interval. *c*–$\searrow e\flat$ is an approximation to such a chord, where the $\searrow e\flat$ is derived by fifteen fifths down. But in minor common chords the condition is that the major third or sixth involved shall be approximately perfect; and this gives the triad *c*–/*e*♭–*g* where the /*e*♭ is derived by nine fifths up. The intermediate form, *e*♭, gives a minor third not quite so smooth as either of the other two; but it is capable of being usefully employed in such combinations as the diminished seventh, and it is preferred by many listeners, as deviating less from the ordinary equal temperament note, from which it has only the departure due to three fifths down. The interval between the harmonic seventh on the dominant and the minor third of the elevated form on the keynote, is the smallest value of the whole tone which occurs, the departure from E. T. of such a tone being due to twenty-two fifths, or about two commas; and although two chords, involving these notes in succession, may each be perfectly harmonious, the sequence is generally offensive to ears accustomed to the equal temperament. Example:—



Custom makes such passages sound effective, especially when the succession is slow enough to enable the ear to realise the fineness of the chords.

Major Third.—This interval has been already discussed; the note taken is that formed by eight fifths down.

Fourths and *Fifths* need no remark.

Depressed Form of the Dominant.—When the dominant is used in such a combination as the following:—

it must be formed by eleven fifths down from the keynote, unless we regard the keynote as changed for the moment, in which case, by elevating the subdominant, we may retain the fifth in its normal position. The most judicious course depends on whether the fifth is suspended or not. Thus, if the fifth is suspended, we may write:—

For if the subdominant be *f*, its third must be *a*, and its sixth must be *d*; *g* then makes a fourth with *d*, which is unbearable to the ear; the fourth must be made correct, and the ways of doing so are shown above. The difficulty may be otherwise got over by writing the passage :—

&c.

Minor Sixth.—This interval is pretty sharply defined. The usual form is /*a*♭, which is got by eight fifths up ; the keynote forms an approximately perfect third with this note by inversion.

Major Sixth (*c*–*a*).—This interval is, as a matter of fact, more sharply defined than one would expect on a first consideration, but we immediately see the reason. Regarded as an inversion of the minor third (6 : 5), it has the ratio (3 : 5), and therefore arises by the interference of the fifth harmonic (tierce) of the lower note with the third (twelfth) of the upper. If c_1 be the lower note, the interference is then on e_3, and the total intensity of the harmonics concerned (3 and 5) is greater than that of the pair (4 and 5) which determine the major third. This interval, then, must be kept strictly to its best value. The *a* is got by nine fifths down.

In chords formed of a succession of minor thirds, major sixths frequently occur. Care must be taken to dispose them so as to make this interval correct. If a deviation is necessary, it is better, if possible, to extend the interval by an octave; the resulting major thirteenth (3 : 10) is not very sensitive.

Minor Seventh.—There are three forms of the minor seventh. To fundamental *c* these are /*b*♭, *b*♭, and *b*♭ :—

/*b*♭ ; ten fifths up ; the minor third to the dominant.

b♭ ; two fifths down ; the fourth to the subdominant.

b♭ ; fourteen fifths down ; approximation to the harmonic or natural seventh.

The two first speak for themselves.

If we compute (Rule 1, paragraph 2) the departure of the harmonic seventh (7 : 4), we find that it falls short of ten semitones by ·31174, or (as we phrase it) the departure is – ·31174, or about one-third of an E. T. semitone. In fact, it is well known that, if we flatten a minor seventh by some such quantity, we can obtain a smooth combination, free from beats. Now the departure of fourteen perfect fifths down is $14 \times \cdot 01955 = - \cdot 27370$; and a note having this departure differs from the natural seventh only by ·03804, or less than the twenty-fifth part of a semitone. Helmholtz pointed out this approximation for the first sime, as far as the writer is aware.

Rule.—The natural or harmonic seventh on the dominant must not be suspended, so as to form a fourth with the keynote.

For the harmonic seventh to dominant *g* is *f*; and *c*–*f* forms a fourth a comma flat, approximately. In ratios this stands as follows : the ratio of tonic to dominant is 4 : 3 ; dominant to its harmonic seventh, 4 : 7; whence ratio of harmonic seventh of dominant to tonic is 21 : 16, or 63 : 48. But ratio of fourth to tonic is 4 : 3, or 64 : 48; whence this fourth differs from the harmonic seventh to dominant in the ratio 64 : 63, or by more than a comma.

Major Seventh.—There is only one form of major seventh which can be used in harmony, viz., ↘*b*; this note is got by seven fifths down; it forms a major third to the dominant. In unaccompanied melody the form *b* produces a good effect. This is got by five fifths up with perfect fifths. It forms a dissonant, or Pythagorean third, to the dominant. The resulting semitone is less than the E. T. semitone by nearly $\frac{1}{10}$ of a semitone.

31. Negative Systems.

Second of the Key.—Two fifths up. The double form does not appear.

Minor Third.—Three fifths down.

Major Third.—Four fifths up.

Fourth.—One fifth down.

Fifth.—One fifth up.

Minor Sixth.—Four fifths down.

Major Sixth.—Three fifths up.

Minor Seventh.—Here we have the only case of a double form. *b*♭, two fifths down, makes a fourth to the subdominant; and ↗*b*♭, ten fifths up, gives the approximate harmonic seventh. This approximation is very close in the best negative systems. Thus, in the negative system of perfect thirds, where $-\cdot 13686$ is the departure of four fifths up, the departure of ↗*b*♭ is $-\frac{10}{4} \times \cdot 13686 = -\cdot 34215$, which exceeds the departure required ($-\cdot 31174$) by $\cdot 03041$, or about $\frac{1}{33}$ of a semitone. The rule about not suspending the dominant harmonic seventh into a tonic fourth holds also in these cases.

Major Seventh.—Five fifths up.

In negative systems the notation marks are commonly omitted. But as the harmonic seventh would thus be written *a*♯, they must be introduced for this or any similar purpose.

As the writer has not yet had in his hands for study an example of a negative system, he is not able to speak of these with the same experience as of positive systems.

32. Number of Units in the Intervals of the Scale.

In Regular Cyclical Systems, to find the number of units in any interval in the scale. Let x be the number of units in the seven-fifths semitone.

$$\text{Then } x\frac{12}{n} = 1 + 7\delta = 1 + 7\frac{r}{n}$$
$$\text{or } x = \frac{n + 7r}{12}$$

It is easy to see that x will always be integral, if the order condition is satisfied (Theorem iii.)—viz., if $7n+r$ is a multiple of 12.

For then, $7(7n + r) = 49n + 7r$; whence, casting out $48n$, $n + 7r$ is a multiple of 12. We can now determine the remaining intervals in terms of x and r:—

In the system of 53, $x = 5$ $r = 1$
In the system of 31, $x = 2$ $r = -1$

The following table gives the general expressions for positive and negative systems, and the numbers for the systems of 53 and 31:—

	Number of Units.			
Interval.	Positive Systems.	System of 53.	Negative Systems.	System of 31.
5-fifths semitone	$x - r$	4	$x - r$	3
Minor tone, 10-fifths tone	$2x - 2r$	8		
Major tone, 2-fifths tone	$2x - r$	9	$2x - r$	5
Minor third	$3x - r$	14	$3x - 2r$	8
Major third	$4x - 3r$	17	$4x - 2r$	10
Fourth	$5x - 3r$	22	$5x - 3r$	13
Fifth	$7x - 4r$	31	$7x - 4r$	18
Minor Sixth	$8x - 4r$	36	$8x - 5r$	21
Major Sixth	$9x - 6r$	39	$9x - 5r$	23
Harmonic Seventh	$10x - 7r$	43	$10x - 5r$	25
Minor Seventh	$10x - 6r$	44	$10x - 6r$	26
Major Seventh	$11x - 7r$	48	$11x - 6r$	28
Octave	$12x - 7r$	53	$12x - 7r$	31

The $-r$'s in negative systems are, of course, positive quantities.

33. Concords of Regular and Regular Cyclical Systems.

The theory which has been established permits us to calculate the departures and errors of concords in the various Regular and Regular Cyclical Systems.

There is another quantity which may be also conveniently taken into consideration in all cases—viz., the departure of 12 fifths of the system. We will call this Δ, putting $\Delta = 12\delta$ where δ is the departure of one fifth.

We have then the following table of the characteristic quantities for the more important systems hitherto known. The value of the ordinary comma $(\frac{81}{80})$ is ·21506. It is comparable with the values of Δ, and if introduced in its place in the table would give rise to a regular non-cyclical system, lying between the system of

53 and the positive system of perfect thirds, the condition of which would be that the departure of 12 fifths = a comma.

Name or n	Order r	$\Delta = 12\delta$ or $12\frac{r}{n}$	Error of Fifth $\delta - \cdot01955$	Error of Third $\cdot13686 - 8\delta$	Error of Harmonic Seventh $\cdot31174 - 14\delta$
17	1	·70588	·03927	−·33373	−·51178
29	1	·41379	·01493	−·27586	−·17101
41	1	·29268	·00484	−·19512	−·02970
Perfect Fifths	—	·23460	—	−·01954	·03804
53	1	·22642	−·00068	−·01409	·04758
Positive Perfect Thirds	—	·20529	−·00244	—	·07223
118	2	·20339	−·00260	·00127	·07445
65	1	·18462	−·00417	·01378	·09635
($\delta = \frac{r}{n}$ is here negative)				13686 + 4δ	·31174 + 10δ
43	−1	−.29707	−·04431	·03784	·06418
31	−1	−·38710	−·05181	·00783	−·01084
Mean Tone. Negative Perfect Thirds	—	−·41058	−·05376	—	−·03041
50	−2	−·48000	−·05955	−·02314	−·08826
19	−1	−·63158	−·07218	−·05367	−·21458

Supplementary Table, exhibiting the Derivation of the Departures of Fifths and Thirds from Equal Temperament, in Terms of Equal Temperament Semitones:—

Name or n	Order r	Departure of Fifth = δ	Departure of Third
17	1	$\frac{1}{17}$	$-\frac{8}{17}$
29	1	$\frac{1}{29}$	$-\frac{8}{29}$
41	1	$\frac{1}{41}$	$-\frac{8}{41}$
Perfect Fifths	—	$\frac{1}{51\cdot151}$	$-\frac{8}{51\cdot151}$
53	1	$\frac{1}{53}$	$-\frac{8}{53}$
Positive Perfect Thirds	—	$\frac{1}{58\cdot454}$	$-\frac{8}{58\cdot454}$
118	2	$\frac{1}{59}$	$-\frac{8}{59}$
65	1	$\frac{1}{65}$	$-\frac{8}{65}$
43	1	$-\frac{1}{43}$	$-\frac{4}{43}$
31	−1	$-\frac{1}{31}$	$-\frac{4}{31}$
Mean Tone. Negative Perfect Thirds .	—	$-\frac{1}{29\cdot227}$	$-\frac{4}{29\cdot227}$
50	−2	$-\frac{1}{25}$	$-\frac{4}{25}$
19	−1	$-\frac{1}{19}$	$-\frac{4}{19}$

34. Application of the Principle of Symmetrical Arrangement to Positive Systems.

This has been sufficiently illustrated by the symmetrical arrangement of the notes of General Thompson's enharmonic organ, given in the writer's previous paper (p. 16).

35. Application of the Principle of Symmetrical Arrangement to Negative Systems.

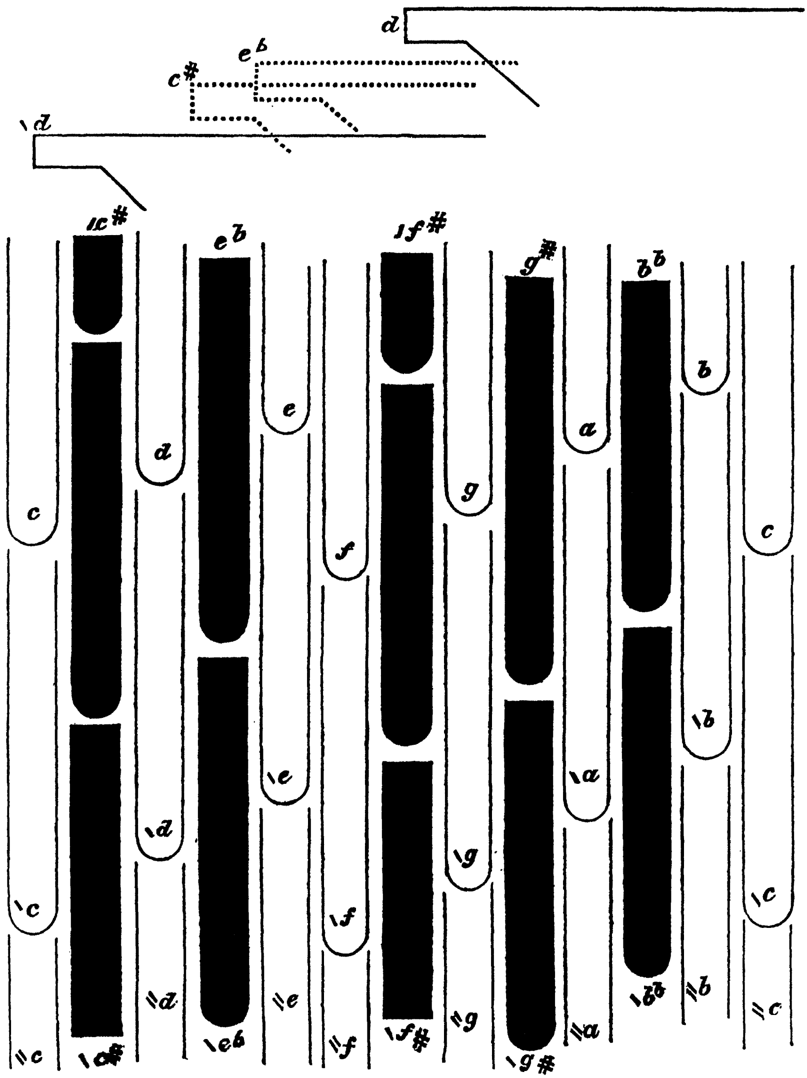
According to the enunciation of the principle, the positions should, in negative systems, be taken downwards as we ascend in the series of fifths; for the departures thus obtained are negative. But it is practically more convenient to use the positive form in negative systems as well. We speak of this form—in which sharp departures are set off downwards, and flat departures upwards—as the 'reversed' form, in contradistinction to the form used in positive systems, which is spoken of as the 'direct' form.

36. Application of the Principle of Symmetrical Arrangement to a 'Generalised Keyboard' for Regular Systems.

A keyboard has been constructed, on the principle of 'symmetrical arrangement,' in the following manner. The octave is taken = 6 in. horizontally: (in ordinary keyboards the octave is $6\frac{1}{2}$ in.) This is divided into 12 equal spaces, each $\frac{1}{2}$ in. broad, and these are called the 12 principal divisions of the octave. A horizontal line gives the positions of an E. T. series where it crosses them all. The keys are placed at vertical and horizontal distances from the E. T. line corresponding to their departures, on the supposition that the arrangement is positive. The departure of 12 fifths up corresponds to a horizontal displacement of 3 in. from the player, and a vertical displacement of 1 in. up. These displacements are divided equally among the fifths to which they may be regarded as due—i.e., the displacement of *g* with respect to *c* is, $\frac{1}{4}$ in. back, and $\frac{1}{12}$ in. up; so of *d* with respect to *g*, of *a* with respect to *d*, and so on.

Although only 3 in. of each key are thus exposed on a plan, yet the keys are all made to overhang $\frac{1}{2}$ in.: and thus the tangible length of each key is $3\frac{1}{2}$ in.

The accompanying figure (p. 136) shows a small portion of the keyboard, on a reduced scale. The keys are each $\frac{3}{8}$ in. broad, and their centres are $\frac{1}{2}$ in. apart. There is thus $\frac{1}{8}$ in. free between the adjacent surfaces of each pair of keys, and $\frac{5}{8}$ in. altogether between the two keys which rise on each side of any given key. This is of importance: e.g., in the chord *c*_*e*_*g*_*c*, taken with the right hand, the first finger has to reach *e* between the adjoining keys *e*♭_*f*, and under the overhanging *e*. The keys in the five principal divisions which have 'accidental' names (e.g., *c*♯ or *d*♭) are black—the rest white.


d
e♭
c♯
\d
\\c♯
e♭
\f♯
g♯
b♭
c
d
e
f
g
a
b
c
\c
\d
\e
\f
\g
\a
\b
\c
\\c
\c♯
\\d
\e♭
\\e
\\f
\f♯
\\g
\g♯
\\a
\b♭
\\b
\\c

37. Application of the Positive System of Perfect Thirds to the 'Generalised Keyboard.'

Helmholtz's System—Just Intonation.

If the thirds such as $c{-}{\searrow}e$ are made perfect, and the fifths flat by ·00244, a quantity which escapes the ear, we have the system here mentioned. Helmholtz makes a mistake in describing it; (Die Lehre von den Tonempfindungen, Ed. 3, p. 495); he supposes that the fifths are sharp, instead of flat, by the above interval; it is easy to see from the context that this is a mistake.

38. The Small Enharmonic Organ—Positive Stop.

The small enharmonic organ, exhibited when this paper was read, contains a stop tuned according to the last-mentioned system. The compass of the 'generalised keyboard' is 3 octaves, from tenor *c* to *c* in alto, and there are 48 notes in every octave. General Thompson's organ, it will be remembered, had 40. Mr. Ellis has pointed out to the writer, that Helmholtz did not employ this system, strictly speaking, but a system of unequally just intonation. The process employed in the enharmonic organ is not, in practice, perfectly equal; but the error is not sensible to the ear. The process is as follows:—The highest note in the series of fifths may be called ${}_1f$. The eight highest notes on the keyboard are tuned by perfect fifths. At the ninth note, in descending, we have the check ${}_1f{-}d{-}{}_1b\flat$. It is then sometimes necessary to distribute a small error among the fifths already tuned, to get the *d* perfect, but this is easy; afterwards all the notes are tuned as perfect thirds in triads; and it is found that the notes so tuned always satisfy the condition of making the fifths sensibly perfect. Of course the fifths are not really perfect, but the imperfection is too small to estimate directly; and it appears that in tuning by triads the positions are assigned in the most correct manner possible.

The action is that known to organ-builders as 'direct'—i.e., the stop-sliders run parallel to the movement of the stop-handles, while the grooves of the soundboard are parallel to the front of the instrument. The action is led from the keyboard, by square and tracker work, to a roller-board of four tiers, which extends to a considerable depth back from the keyboard. The rollers are parallel to the grooves, and to the front of the instrument, and are a little longer than the width of the keyboard. By these means the whole series of action is reduced to two long rows of pulldowns, along the two sides of the windchest, and thus every note is easily accessible for regulation, and all the pallets can be got at with ease. This form of arrangement is susceptible of indefinite extension, and is that which must be adopted in the event of its being desired to build an instrument of the kind with several stops. This plan, the general features of which were devised by the writer, has been most ably carried out by Mr. T. A. Jennings.[1] The keyboard, the tracker-work, and the roller system specially invite attention by their exquisite workmanship.

39. Improved Screw Stoppers for Tuning Pipes.

The enharmonic organ contains an improvement, also devised by the writer, and carried out by Mr. Jennings. Stopped diapason pipes of metal were selected, as occupying less space than any others; the proverbial difficulty of managing the tuning of such pipes became the subject of consideration, and the screw stoppers here used were devised. The essential feature is, that the stopper contains a moveable slider inside, which is worked by a screw from without. When the stopper is once approximately placed, the body of it never needs to be moved again, and the accurate tuning required for the writer's purposes is easily and conveniently performed by means of the projecting screw.

40. Application of the Notation to the System of 53.

The notation introduced for positive systems (paragraph 9) is susceptible of various accessory rules, according to the system it is attached to. It is required to find rules of identification for passing from one principal division of the octave to another in the system of 53.

Rule.—In the system of 53 the notation becomes subject to the following identifications :—

If two notes in adjoining principal divisions (e.g., *c* and *c*♯) be so situated as to admit of identification (e.g., a high *c* and a low *c*♯), they will be the same if the sum of the elevation and depression marks = 4; unless the lower of the two divisions is black (accidental), then the sum of the marks of identical notes = 5.

This can only be proved by enumeration of a case in each pair of divisions. The mode of verification in any case is the following:—

Noting that the 5-fifths semitone is four units (scheme following Theorem iii.) we see that *c*–*c*♯ is four units, whence ////*c*–*c*♯, ///*c*–*c*♯, //*c*–*c*♯ are identities; or, again, *c*♯–*d* is four units, and ////*c*♯–*d*, ///*c*♯–*d*, are identities.

[1] Organ-builder (address at Mr. Fowler's, 127, Pentonville Road). Mr. Jennings has built an ordinary organ for the writer, and also the large enharmonic harmonium.

41. APPLICATION OF THE 'GENERALISED KEYBOARD' TO THE SYSTEM OF 53.

The Enharmonic Harmonium.

The practical studies on positive systems, which the writer has had the opportunity of making, have all been conducted with the assistance of a large harmonium with a very extensive 'generalised keyboard,' containing eighty-four keys in every octave, tuned in accordance with the system of 53. This was also built by Mr. Jennings. It has now been completed about two years. The arrangement is as follows :—

The note $\diagdown\diagdown\diagdown c$ is taken as the first note of the series, and receives the characteristic number 1. Then c is 4, and the remaining numbers can be assigned by the rules for the identifications in the system of 53 given above.

A number of notes at the top of the keyboard are thus identical with corresponding notes in the adjacent principal divisions on the right at the bottom,—e.g. $\diagup\diagup c = 6 = \diagdown\diagdown c\sharp$. These permit the infinite freedom of modulation which is the characteristic of cyclical systems. For in moving upwards on the keyboard, we can, on arriving near the top, change the hands on to identical notes near the bottom, and so proceed further in the same direction, and *vice versâ*. It is to be noted that, in positive systems, displacement upwards or downwards on the keyboard takes place most readily, by modulation between related major and minor key—not, as has been commonly assumed, only by modulation round the circles of fifths. In negative systems, on the contrary, displacements take place only by modulations of the latter type. Suspensions of the harmonic seventh, however, give rise to rapid displacements in both classes of systems.

The keyboard of the harmonium contains seven tiers of levers, the variations in the position of the notes of each tier being determined by the patterns of the keys attached to them. Each of these tiers communicates, through a row of squares, with a row of horizontal stickers. The windchest is vertical, and the valves are ranged on it in seven horizontal rows. The valves have small tails attached, and the stickers open the valves by pressing on the tails. There is no attachment between the stickers and the valves.

The original object of this arrangement was, that the windchest being easily lifted out, another might at pleasure be substituted, containing a different system of tuning. The two additional windchests which were constructed were, however, accidentally spoiled, and have never been replaced. The great practical interest of the application of the mean-tone system to the generalised keyboard leads the writer to hope that he may be able, at some future time, to provide an additional windchest for this system.

42. Application of the System of 118 to the 'Generalised Keyboard.'

The 5-fifths semitone is here nine units, and the 7-fifths semitone is eleven units. The major tone (2-fifths tone) is consequently twenty, and the minor tone (10-fifths tone) is eighteen. Hence the notes in the successive principal divisions are alternately odd and even, and the identifications lie in alternate columns. These are not here further investigated, as no practical use has been made of the system. If $c = 1$, $c\sharp = 10$, ${}_1c\sharp = 12$, $d = 21$. It would be possible to construct a keyboard on the principles already explained, which would give complete control over the notes of the system of 118. A portion of such a keyboard would be practically undistinguishable from one tuned to the positive system of perfect thirds, as the error of the thirds of the system of 118 is too small to be perceived by the ear.

43. Notation for Negative Systems.

As negative systems correspond, in the formation of their thirds, to the ordinary usage of musicians (paragraph 7), the notation adopted for the purpose of indicating position in the fundamental series of fifths becomes to a certain extent unnecessary; for in these systems the distinction between $a\sharp$ and $b\flat$ is a true and essential one, and is sufficient, within a certain limited range, without the introduction of the notation marks. These will, however, continue to be occasionally required. For instance, in the case of the harmonic seventh to c, it would be intolerable to have to write it $a\sharp$, and yet this would be its correct designation. It seems better, therefore, to retain the notation to this extent, that wherever it is used it overpowers, so to speak, the meaning of the flats or sharps employed, and denotes derivation in the series of fifths, just as in the case of positive systems; we can then write the above note ${}_1b\flat$.

In the application of the keyboard to negative systems, a sharp always signifies the higher on the board of the two notes, between which choice may be made; and it must be remembered that the higher on the board is, in this case, the lower in pitch. But whenever the notation appears, it overpowers the above rule, and acts precisely as in positive systems.

44. Application of the Negative System of Perfect Thirds (Mean Tone System) to the 'Generalised Keyboard.'

Small Enharmonic Organ—Negative Stop.

If the thirds such as c–e are made perfect, and the fifths ·05376 (or a quarter of a comma) flat, we have the mean-tone system. The forms on the keyboard of scales and chords in negative systems are different from those in positive systems. The scales are very easy to play, and the chords also. It is expected that this application may prove of practical importance. By the employment of reversed symmetrical arrangement, we can use the same keyboard as that for positive systems.

Following the scale of unmarked naturals on the plan of the keyboard, we can realise the nature of the fingering. It is the same as that of the Pythagorean scale with the system of perfect fifths on the keyboard. The tones are all 2-fifths tones, and the semitones both 5-fifths semitones.

The small enharmonic organ exhibited to the Association contains, on the three lower tiers of its keyboard, a second stop, the pipes of which are tuned to the above system. It may be regarded as a great extension of the old unequal temperament—that is to say, the performance everywhere produces the same results as if the best chords of the old unequal temperament were employed.

45. Application of the Negative System of 31 to the 'Generalised Keyboard.'

The objection to the last-mentioned arrangement (mean-tone system) is, that a certain rather considerable amount of modulation would carry the performer beyond the limits provided; this may be overcome by substituting for the mean-tone system the Huyghenian system of 31, the intervals of which differ very little from those of the mean-tone system. The thirds, instead of being perfect, would be ·00783 sharp, or considerably less than the 100th of a semitone; and the fifths ·05181 flat, instead of ·05376; difference = ·00195, or about the 500th of a semitone.

The application of the principles of symmetrical arrangement (reversed) to the numbers of the series of 31 is easily deduced from the table of the numbers of units in the intervals of that system. The five tones of any scale consist each of five units—the two semitones each of three. It is only necessary to remember, that the lower numbers of the series always stand above, and the higher numbers of the series below, in each principal division of the octave, according to the definition of a 'reversed' symmetrical arrangement.

46. Tuning.

It was the writer's intention to have introduced at this point an essay on certain improved methods of tuning. These will depend mainly on the employment of an extremely accurate form of metronome, devised many years ago by Scheibler,[1] for the

[1] The Scheibler metronome was exhibited at the meeting, finished all except the graduation. It has been constructed for the writer by Messrs. Tisley & Spiller, 172, Brompton Road.

counting of beats. On account partly of the difficulty experienced in getting the metronome constructed, and also of the extent of the subject generally, it is preferred to treat it at some future time in a separate communication. An account will be given now only of the mode in which the system of 53 was actually tuned on the harmonium.

The standard *c* was taken to have 512 vibrations per second; this was obtained by comparing several Philharmonic forks professing to have this pitch. Although these agreed sufficiently well with each other, it appeared, subsequently, that the resulting *c* was not satisfactorily derived, and much inconvenience was experienced during the tuning from this cause. The processes then within reach were not sufficiently accurate for a perfect independent determination of the standard pitch. It will be seen, when the subject of Tuning is treated, that an independent determination of this element is always necessary; and this can be made with great accuracy by means of a few pipes or reeds, and the improved metronome.

It was first necessary to produce a set of 53 reeds of approximate accuracy, for the guidance of the builder. A small tuning-machine was constructed, which contained an organ-bellows and two windchests—one for 12 organ-pipes, the other for 12 harmonium reeds. The pipes were fitted with caps and handles to facilitate the tuning. But it was at once discovered, contrary to expectation, that the process of tuning the reeds was far more delicate, certain, and easy than that of tuning the pipes; and accordingly the pipes were never used at all. A table was then calculated, showing the number of beats per minute that must be made by each consecutive pair of notes of the system of 53, in the tenor octave. A set of reeds was then tuned by this table, the beats being counted with a watch, and the practically perfect fifths of the system employed as checks. A very fair result indeed was obtained in this way with but little difficulty. But the following weak points prevented the adoption of the process as an exact one:—

In the first place the counting of the beats by the watch proved an extremely troublesome and rather inaccurate method. This must be replaced by the improved metronome.

Secondly, as the reeds were tuned on the machine, when they came to be removed, and placed on their seats in the harmonium, they underwent small alterations; and although the result on the machine was pretty good, it was no longer sufficient when the reeds were transferred.

Another process, of a more delicate nature, was therefore adopted, by which the dependence on the counting was minimised. This was also first carried out on the tuning-machine; and, before explaining the process, it will be well shortly to describe the arrangement of the machine, as far as the reeds are concerned.

The reed windchest consists of a box, which receives the wind from the bellows through an aperture closed by a valve moveable from outside. On the top of the box the lid fits air tight; it opens on hinges, and is fastened down, when closed, by two handscrews; it can be opened or closed in a few seconds. The lid is pierced by slits, outside which are small valves, and handles for opening the valves; and about the end of the slits there is, inside the lid, a peculiar arrangement of winged screws, of a form which can be got from any ironmonger: these enable any reed to be firmly secured over one of the slits in a few moments. It is essential that the pressure of the screws shall not seriously deform the case of the reed. If this happens the pitch is altered. It is easy to test whether this happens, by tuning two reeds to a fifth, which is the most sensitive interval, and then screwing down one of them a little tighter; the slightest alteration of pitch is an indication that the screws are in fault. Stout handscrews were first employed; and it was not for some time that the considerable error they caused was detected. No results of any value could be obtained with them. The chief practical point appears to be to have the screws as near as possible to the corner of the reeds. When the lid of the box is turned over, the reeds are before the tuner, as if fixed on a table, in a convenient position for the use of the tools. The box is about 2 feet long, 4 inches broad, and 2 inches deep. The small dimensions facilitate the production of difference tones, and these are heard with great clearness. The following scheme for the system of 53 depends on difference tones, and accordingly it was executed with ease on the machine.

One point which must be noted before passing to this is, the pitch of reeds varies very considerably with the strength of wind employed. It is therefore found best to do away with the use of the bellows-valve altogether, and use light pressures on the bellows. It is always possible to find out whether a reed is too high or too low; for if the slit-valve over the reed is partially closed, its pitch falls, and it can be perceived whether the beats get better or worse.

In the system of 53 the fifths are taken to be perfect. The thirds, however, are perceptibly flat. If a major triad is sounded, the displacement of the third gives rise to displacements in opposite directions of the difference tones formed with the other notes; and it was by the beats of these displaced difference tones that the thirds were tuned, in the first part of the bearings. These thirds were tuned either in the position c_2–$\searrow e_2$–g_2 (between fifth and fundamental), or in the position g_1–c_2–$\searrow e_2$ (above a fourth). The beats in the latter case are half the number of those in the former, as in the latter only one of the difference tones is displaced by the error of the third. If the position were $\searrow e_1$–g_1–c_2, the particular beats to be observed would not arise at all. The principles explained in paragraph 27 suffice for the calculation of the numbers required.

The first part of the following system of bearings was carried out with ease on the tuning-machine; but when it was repeated on the harmonium, it was found that the difference tones were so faint as only to be heard with difficulty, no doubt owing to the great size of the windchest. The method is not, therefore, recommended for adoption in such cases. With the class of pipes employed in the small enharmonic organ the difference tones are strong, and this method might be expected to be successful:—

Bearings of the System of 53.

$c_2 = 4 = 512$ vibrations.

First Part.

Tune from	Fourth or Fifth.		Third Flat.	Beats per minute.
c_2. 4	f. 26		\ a_2. 43	42
\ a_2. 43	\ d_2. 12		\ f_2♯. 29	69
\ f_2♯. 29	\\ b_1. 51		\\ d_2♯. 15	58
\\ e_2 b 15	\\ a_1 b 37		\\\ c_2 / b_1 . 1	48
/ b_1. 1	/ e_1. 23		/ g_1♯. 40	40
/ a_1 b 40	/ e_2 b 18	check	c_2. 4	50

We thus return in a circle to the standard c: even without counting the beats, we can easily distribute the small error amongst these few steps, so that it shall be nowhere prominent. In fact, this is a less difficult process than laying the bearings of the ordinary equal temperament. In counting the beats, it is only necessary to notice beforehand what the pitch of the difference tone will be; by sounding this note lightly first, so as to direct the attention to it, it is easy to separate its beats from those of the tierce and the others present, if the difference tones are of sufficient strength to be heard at all. We then proceed to fill up the gaps with perfect fifths, distributing any error amongst the six or seven fifths of each gap:—

Second Part.

Bearings of the System of 53.

Tune from	Fourths or Fifths.				
f_2. 26	b_1 b. 48	e_2 b. 17	a_2 b. 39	d_2 b. 8	Distribute
\ a_2. 43	\ e_2. 21	\ b_1. 52	—	f_2♯. 30	any error.
\ d_2. 12	\ g_1. 34	\ c_2. 3	\ f_2. 25	\ b_1 b. 47	
\ f_2♯. 29	\ c_2♯. 7	\ a_2 b. 38	—	\ e_2 b. 16	
\\ b_1. 51	\\ e_1. 20	\\ a_1. 42	\\ d_2. 11	\\ g_2. 33	
\\ e_2 b. 15	\\ b_1 b. 46	\\ f_2♯. 24	—	\\ c_2. 2	
\\ a_1 b. 37	\\ c_2♯. 6	\\ f_2♯. 28	—	\\\ b_1. 50	
\\\ c_2. 1	\\\ g_2. 32	\\\ d_2. 10	\\\ a_2. 41	\\\ e_2. 19	
/ e_1. 23	/ a_1. 45	/ d_1. 14	/ g_2. 36		
/ e_2 b. 18	/ b_1 b. 49	/ f_2. 27	/ c_2. 5		
/ a_1 b. 40	/ c_2♯. 9	/ f_2♯. 31	—	b_1. 53	
c_2. 4	g_2. 35	d_2. 13	a_2. 44	e_2. 22	

This process was gone through many times on the harmonium, after all the reeds were in position. The alterations that appeared at first to take place in the reeds in a few days, were considerable, and it was not for some time that it was perceived that it was the action that was in fault, and not the reeds; small variations in the action affect the pitch of the reeds considerably. But since the source of these disturbances was realised, the reeds have been left alone, and on screwing up the action again at any time, the tuning has always been found to leave little to be desired. Two years have elapsed since the reeds were last touched, and about a year since the action has been touched; and the instrument is now in first-rate order.

47. The tuning of the principal negative system (mean-tone or mesotonic system, negative system of perfect thirds, extension of old unequal temperament) was performed as follows on the small enharmonic organ. Starting from the lowest middle c ($\searrow c$) on the keyboard, $\searrow g$ was tuned temporarily a perfect fifth, and then $\searrow e$ accurately a perfect third; $\searrow g$ was then flattened a little, and $\searrow d$ and $\searrow a$ tuned successively a little lower than consonance would require. Check, $\searrow a - \searrow e$ must be a fourth or fifth of the same quality with the others. After a little practice this was easily carried out. Every succeeding set of 4 fifths was tuned in the same way, checking always by a third tuned in a perfect triad.

48. Higher Approximations.

For an outline of the theory of multiple systems, and of certain systems of higher orders, which are not of much practical interest, reference is made to the writer's paper on 'Temperament,' in the current volume of the Proceedings of the Royal Society.

49. Complete Symmetrical Arrangement.

This form of higher approximation is of some interest, as admitting of a very clear representation of the accurate relation between perfect fifths and thirds—also as introducing a form of symmetrical arrangement, which, although the most complete, is also the most simple in its origin.

Let a series of E. T. notes be placed, in order of the scale, at equal distances on a horizontal line; if we represent each note by a dot, we shall have one of the horizontal rows of the table at page 16. Let other E. T. scales be placed above and below the first, each note of each successive row being higher than the corresponding note of the next row below by the departure of a perfect fifth ($=\cdot01955$). Then we have a symmetrical arrangement of perfect fifths, with all the positions filled up—*i.e.*, between c and $\nearrow c$ there are 11 notes and 12 intervals, each $= \cdot01955$. Now $\searrow e$ was the third got by 8 fifths down, and it is a little too low. If we take the *e* which is seven steps below the E. T. row containing *c*, instead of that which is eight steps below, we obtain a third the error of which is little more than ·00001, or one hundred-thousandth of an E. T. semitone. In fact,—

Departure of seven fifths	=	·13685 00606
„ third	=	·13686 28614
error		·00001 28008

carrying the calculation to ten places. By adjusting the exact value given to the difference between the rows, it is possible to get the approximation more exact, but this is of no material interest. It would be possible, if desired, to construct a keyboard for such a system, the dots of the figure of paragraph 13 being replaced by concertina-keys. It would then be possible to play fifths and thirds perfect to the above degree of approximation.

50. Application of Regular Cyclical Systems to Complete Symmetrical Arrangements.

[The matter of the following section was suggested by the discussion of the properties of the system of 612, the importance of which was pointed out by Captain J. Herschel, F.R.S., to Mr. A. J. Ellis, F.R.S., and by him to the writer.]

In order that any Regular Cyclical System may be applicable to a complete symmetrical arrangement, it must be a multiple of the ordinary equal temperament, or n must be divisible by 12.

For the essence of a complete symmetrical arrangement is, that it may be regarded as made up of a number of E. T. systems, each of which is represented by a horizontal row of the symmetrical arrangement. (See p. 16.)

Any system which is a multiple of the equal temperament may be also regarded as of the 12th, 24th, . . . order. This follows at once from the Theorems on the orders of systems.

To find a cyclical system which shall possess in all its parts the properties of the complete symmetrical arrangement above described:—

The departure of the perfect fifth may be written $\frac{1}{51\cdot151}$, and the system sought must have fifths closely corresponding with his in departure. If r be the order of the system, we have then $\frac{r}{n} = \frac{1}{51\cdot151}$, or $n = r.\ 51\cdot151$ approximately, where r and n must both be multiples of 12. Let $r = 12$. Then the best value of n is 12×51, or 612. The fifths of this system have the departure $\frac{1}{51}$. Error = ·000,058 sharp. The thirds, being formed, according to the rule for complete symmetrical arrangements, by seven steps down instead of eight, have the departure $\frac{7}{51} = -\frac{8}{\frac{8}{7}\cdot51} = -\frac{8}{58\cdot286}$. Error = ·000,39 flat.

It is easy to see, from the correspondence of these errors, that they may be much diminished by making the fifths more perfect.

The next higher value of r, which gives any considerable approximation applicable to complete symmetrical arrangements, is evidently 240, since $240 \times {\cdot}15 = 36$. Then $n = 240 \times 51{\cdot}15 = 12{,}276$.

The departure of the fifth is $\frac{1}{51\cdot15}$. Error = 3 in the seventh place of decimals; and error of the third = a unit in the fifth place.

51. Practical Use of Positive Systems.

An example of music written for positive systems is appended, as well as a short cyclic modulation in the system of 53 (pp. 151–154). This last can be played on the harmonium, but not on the small enharmonic organ. The principal points in the harmony of these systems which have struck the writer occur in the example. It is to be specially noticed how certain forms of suspension have to be avoided—partly because they produce dissonances, partly because they occasion large displacements up and down the keyboard. The result of the writer's practical experience is, distinctly, that there are many passages in ordinary music which cannot be adapted with good effect to positive systems; and that the rich and sweet masses of tone which characterise these systems, with the delicate shades of intonation which they have at command, offer to the composer a material hitherto unworked. The character of music adapted for these systems is that of simple harmony and slow movement; it is a waste of resources to attempt rapid music, for the excellence of the harmonies cannot be heard. The mean-tone system is more suitable for such purposes.

Some examples of the unsuitability of the positive systems for ordinary music may be here instanced:—

(1.) The opening bars of the first prelude of Bach's 48. The second bar involves the depressed second ($\searrow d$), and in the third bar this changes to d; the melodic effect is extremely disagreeable on the harmonium. It does not strike the ear much with the stopped pipes of the little organ.

(2.) The example of cyclical modulation at page 154, the harmony of which is extended from a passage in the second prelude of Bach's 48.

The effect of this is not bad when one is accustomed to it; but it alters the character of the music completely, and is very disagreeable to unaccustomed ears.

(3.)

The two *g*'s, to which attention is here called by asterisks, illustrate a difficulty of constant occurrence in the adaptation of ordinary music to these systems. The *g* is here required to make a fourth to the depressed second of the key (*d*), and also a fifth to the keynote. But the first condition requires the note *g*, the second *g*, and it is impossible to avoid the error of a comma somewhere. It may be said that the first *g* is only a passing note; but with the keen tones of the harmonium such dissonances strike through everything, even on the least emphasised passing notes. Although the second *g* seems to the writer to be legitimate, it would be intolerable on the harmonium. The smoother tones of the organ render such effects less prominent.

(4.) The third phrase of a well-known chant:—

To keep in the key of *f*, the *g* should fall to *g* at the second chord; but this direct descent on the suspended note would sound bad—consequently, the whole pitch is raised a comma at this point by the suspension; and the chant concludes in the key of /*f*, as it is not possible anywhere to descend again with good effect. This would be inadmissible in practice, as the pitch would rise a comma at each repetition. The resources of the system of 53 admit of the performance of the repetitions in this manner, but the case is one in which the employment of this effect would be unsuitable.

On the organ it might be possible to take the last chord written above *g* — *d* — *g* — *b*♭, which would get rid of the difficulty. On the harmonium, however, this drop from the minor chord of *g* to that of *g* is inadmissible.

In the example of music written for the positive systems, it is to be noted that the notation-marks are used as signatures, exactly as flats and sharps are in ordinary music. The sign adopted for neutralising them is a small circle (o), which is analogous to the ordinary natural. If the general pitch had to be raised or depressed by a comma, the elevation or depression mark would be written large over the beginning of the staff:—

Several points in the harmony are regarded as experimental. For instance, in the inversion of the dominant seventh with the seventh in the bass, the employment of the depressed (harmonic) seventh has on the harmonium an odd effect; although, when the chord is dwelt on, it is heard to be decidedly smoother than with the ordinary seventh. The effect appears less strange on the organ. On this and other points the judgment of cultivated ears must be sought, after thorough acquaintance with the systems.

The following points may be noticed in the example :—

At the beginning of the seventh bar it would be natural, in ordinary music, to suspend the *a* from the preceding chord, thus :—

As, however, the first *a* is *a*, and we are modulating into *g*, whose dominant is *d*, the suspension is inadmissible, as it would lead to the false fifth *d* – *a*.

In bar 14 the ordinary seventh *d* to dominant *e* is employed in the bass instead of the harmonic seventh *d*, so as to avoid the small tone *d* – *c*. The latter has a bad effect in the minor key, as before noticed, and this is specially marked in the bass.

Bar 19.—The use of the tonic as first note in the bass is prevented by the presence of the harmonic seventh on the dominant (paragraph 30).

Bar 24.—This singular change is pleasing in its effect when judiciously used, but it is advisable to separate the two forms of the chord by a rest.

Bar 30.—The smoothness of the approximate harmonic seventh is here applied to the sharp sixth. This effect is the most splendid which the new systems afford; nothing like it is attainable on ordinary instruments.

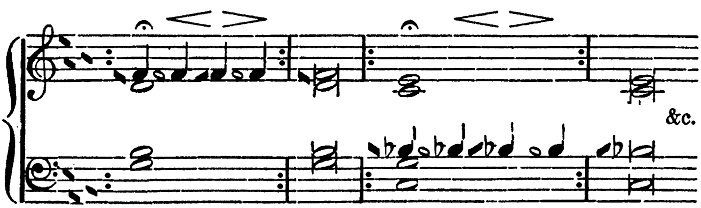
Bars 34 *and* 35.—Here the natural course would be to make the bass :—

the harmony remaining the same. We have, however, arrived at our *d* as the fifth to *g*, and it is not possible to suspend it unless we raise the *a* to *a*. It has not a good effect where a passage is repeated as here, if the repetition is in a slightly different pitch. The suspension is therefore avoided.

Bar 37.—This is a very charming effect. The transient modulation to dominant *d* gives the depressed keynote, *c*, as harmonic seventh.

52. Comma Scale.

The following is an example of a novel effect which is attainable in positive systems. If the chord of the harmonic or natural seventh be sustained, this seventh may be made to rise and fall again through two or more single commas. The effect to unaccustomed ears is disagreeable at first; but the writer has become so familiar with these small intervals, that he hears them as separate notes without the sensation they commonly produce of being one and the same note put out of tune. There can be no doubt that the reception of such intervals is a question of education, just as the reception of semitones was, in the early history of music, a step in advance from the early five-note scales. The following passage, as executed on the enharmonic harmonium, which admits of a swell of the tone, has a dramatic effect:—



53. Series of Major Thirds.

The chord to which attention is called consists of two perfect thirds and the octave. The third $\diagdown g\sharp - c$ has a departure due to 16 fifths up, and an error from the perfect third of about two commas. It may be called the 'superdissonant' third, by analogy from the dissonant or Pythagorean third, which has an error of one comma. We have the choice, if we prefer it, of arranging the chord with two dissonant thirds, thus:—

$$c - \diagdown e - g\sharp - c.$$

The two last thirds are ordinary dissonant thirds; the writer prefers the first arrangement. It is a matter of taste.

The examples were played on the positive stop of the small enharmonic organ, at the meeting of the Association.

54. Mean Tone System.

It has been pointed out, that the generalised keyboard admits of the control of negative systems by means of remarkable simplicity. Not only is the fingering very simple, but the sharps and flats of ordinary music furnish indications, the interpretation of which is clear, the finger going up for the sharp and down for the flat. Although the mean-tone system, on the small organ, was completed only just in time for the meeting of the Association, yet the writer succeeded, at some pains to his ears, in studying the mechanism of that system, to a small extent, on the keyboard of the 53 harmonium. The effect of playing on that instrument in negative form is, of course, to use Pythagorean thirds; and, in consequence, the writer may probably claim to have a greater practical acquaintance with the Pythagorean system than anyone else is likely to possess. The effect is not such as to lead to a desire for its adoption. The examples selected for performance were the first, second, and 33rd preludes of Bach's 48. The fugues were found impracticable, on account of the want of the lower octave; this does not cut so deeply into the preludes selected as to prevent their performance. These three examples were played on the mean-tone stop of the small enharmonic organ at the meeting.

Conclusion.

In the foregoing communication the endeavour has been made to give a simple and general theory of the division of the octave, to provide a system of notation by which the notes of different systems can be expressed for practical purposes, and to solve the mechanical problem for keyed instruments. It may be now affirmed that there is nothing whatever, except study, necessary to enable anyone to obtain a practical command of any system that may be desired; and it is believed that the difficulties of that study have been reduced to a minimum.

Example for Systems of approximately Perfect Fifths, with a Compass of Three Octaves.

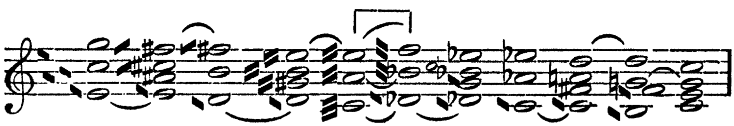
H=Harmonic or Natural Seventh, or inversion thereof.


1mo.
H
7
2do.
H
3
H
14
H
19
H
H
24


H
30
H
H
34
H
H
37
H
H

EXAMPLE OF CYCLIC MODULATION IN THE SYSTEM OF 53; THE HARMONIES TAKEN FROM BARS 5 TO 10 OF THE PRELUDE No. 2 OF THE 48 (BACH); TRANSPOSED AND EXTENDED.

Identifications.



DISCUSSION.

The CHAIRMAN said it was extremely difficult even for experts to follow a paper of this kind when read, but when printed in the Transactions of the Society, which were in active preparation, it would no doubt be perused with great interest. There was not much time left for discussion, but if any member wished to put a question he was quite sure Mr. Bosanquet would be pleased to answer.

Mr. A. J. ELLIS, F.R.S., F.S.A., said he should like to read the few lines which Mr. Bosanquet had referred to as expressing the opinions of Helmholtz with regard to the cause of beats: 'The roughness of a combination of two tones depends, then, in a compound manner on the magnitude of the interval, and the number of beats produced in a second. On seeking for the reason of this dependence, we observe that, as before remarked, the beats on the ear can exist only when two tones are produced sufficiently near in the scale to set the same elastic appendages of the auditory nerve in sympathetic vibration at the same time. When the two tones produced are too far apart, the vibrations caused by both of them at once in Corti's organs are too weak to admit of their beats being sensibly felt.' Then with regard to the experiment with Herr Joachim, he says: 'I was fortunate enough to have an opportunity of making similar observations, by means of my harmonium, on Herr Joachim. He tuned his violin exactly with the *g d a e* of my instrument. I then requested him to play the scale, and immediately he had played the third or sixth, I gave the corresponding note on the harmonium. By means of beats it was easy to determine that this distinguished musician used the depressed *b*, and not the other, as the major third in G, and the depressed *e*, and not the other *e*, as the sixth.'

With regard to the melodic feeling of playing, he might say that the experiments of Cornu and Mercadier, as given in the *Comptes Rendus* (which he had calculated out and reduced to numbers with equal semitones and fractions of them), were so conducted that the pitch was determined by a mechanical process, and not by the ear, so that they might be entirely depended upon. The performers were quite unaware of what was being done, and played as usual; but the result was that in playing harmonies they always used the just intonation, the perfect third and fifth; but in playing melodies they used the intonation, which was supposed to be Pythagorean; at any rate they played the thirds and sixths, and especially the sevenths, sharper than in equal temperament, the seventh being sometimes a comma sharper. He had also observed, in calculating out the numbers, that they seldom played the same interval twice alike, that their fifths and octaves were very frequently changing. In playing the minor scale, on the other hand, the minor third and the minor fifth were taken flatter than they would be even on the Pythagorean method. It was evident, therefore, that the performers had not made up their minds to any particular scale, but they had a general feeling of brightening by taking the intervals sharper, and dulling by taking them flatter; but that when they played harmonies their feeling was to get consonance.

Mr. SALAMAN understood that the practical results sought for by these investigations was to ensure more perfect tuning of musical instruments, rather than to have any effect on the composition of music.

Mr. BOSANQUET said his view most decidedly was, that there was a new material for composers to work with.

Mr. McNAUGHT asked if it were correct to say that existing music could not be performed in just intonation—vocal or instrumental?

Mr. BOSANQUET said much of it could not.[1] No one could execute those small intervals with accuracy by the voice; they could only be examined by an instrument. The voice would very readily adapt itself to the instrumental accompaniment.

Mr. CUMMINGS thought the harmonium (which Mr. Bosanquet had referred to as the instrument by which he had made some of his experiments) was not to be depended upon for anything whatever. It seemed never to give a chord fit to listen to, and therefore he did not think any test of that kind could be relied upon. It gave out so many harmonics that you could not judge of any music fairly by its means.

[1] See paragraph 51 (3), &c.

Mr. Bosanquet said his harmonium had stood perfectly in tune for two years. The reeds were at least twenty times as sensitive as an organ-pipe, and had remained perfectly in tune.

Mr. Cummings remarked that all the harmoniums he had ever tried seemed to give out harmonics more freely than any other instrument. If Mr. Bosanquet's was different in that respect, he should be glad to hear it.

The Chairman observed that the resultant sounds of an harmonium were generally detestable.

Mr. Bosanquet said the great defect in the harmonium was that it was so sensitive to tuning, and that very fact made it so valuable to him in his experiments, because differences which the ear would not detect in another instrument became evident at once on the harmonium. The tuning of an harmonium was far more accurate than that of organ-pipes; you could tune any quantity of the latter from an harmonium, but you could not reverse the process. This fact made it so valuable as a means of research.

The Chairman said he had had the advantage of hearing Mr. Bosanquet's harmonium, and certainly some of the effects were exceedingly beautiful: for instance, its powers were singularly illustrated in a passage commencing the 'Stabat Mater' of Palestrina, which was frequently quoted as an instance of the harshness of the old masters. As Mr. Bosanquet played it, the effect was certainly altogether different—much more delicate, aërial, *spirituelle* than he had ever heard it played. Of course four good singers would unconsciously give the right intonation to it, as was remarked by Burney in his account of the Pope's Chapel, that the singers there unconsciously sing in what is no doubt perfect tune. The point that remained, and the question upon which musicians seemed never to have been satisfied, was whether the mechanical difficulties of obtaining these effects were not so great as to render them useless for practical purposes. As he understood Mr. Bosanquet, this organ required about six times the number of keys and pipes of an ordinary instrument. Now, if the organ of St. Paul's were multiplied by six or seven, it would hardly go under the dome. Besides that, he thought some of the combinations which had been put before them as beautiful were hideous; some of the harmonic sevenths, for instance, were extremely disagreeable.

Mr. Bosanquet said he had always found those notes disagreeable to such persons as had a strong sense of absolute pitch. The only way to like them was to listen to the chords. He never knew anybody who had a good ear who liked them at first.

The Chairman: Then comes the question, what is a good ear, and who has it?

Mr. Bosanquet said he had frequent experience of all sorts of people coming to hear his harmonium, and the result was, that persons with acute ears, but not much musical education, liked the chords, and always picked out the effects which he liked best himself, as the result of long custom; but persons who had the scale firmly in their heads, as no doubt the Chairman had, did not like the departure from the usual value of the notes. They did not think of the consonance at all; the question with them being, not whether it was smooth, but whether it was what they were accustomed to. The question with him was simply one of smoothness.

Mr. CUMMINGS said the chord of the sixth on A♭, at the beginning of the second page of the example which Mr. Bosanquet played, sounded to him very flat indeed.

Mr. BOSANQUET said that was A♭ *raised.* It was a curious fact, that people with highly-educated ears almost always singled out the true minor third of the chord as disagreeable. The true minor third was always raised.

Mr. A. J. ELLIS thought a few lines from 'Barrow's Travels in China' had some bearing on the point of the ear being accustomed to certain intervals. He said the tones were far from being disagreeable, but their construction was so irregular that they did not appear to be reducible to any kind of scale. He went on to say that the Chinese affected not to like the Ambassador's band, which, they pretended, produced no music, but a confusion of noises. It was a very remarkable fact, to be taken in connection with this, that 2,000 years before Pythagoras (according to the Chinese rules, translated by Amyo), the Chinese had a scale of twelve perfect fifths, producing a complete succession; and that in the year 1537, long before it was dreamed of in Europe, a Chinese prince gave the lengths of a series of organ-pipes, producing theoretically perfect equal temperament.

The CHAIRMAN said it was remarkable that those persons who were generally considered the most highly gifted as composers had all been accustomed to the same kind of tuning, and John Sebastian Bach took the trouble to write, for what was obviously the equal temperament, forty-eight preludes and fugues in the different keys. No doubt the present system was theoretically wrong, but it was one which satisfied Mozart, Beethoven as long he could hear, and many others, including all the great singers.

Mr. BOSANQUET desired to remark that he had never advanced the position that the ordinary scale was wrong.[1] No doubt it was perfectly good in practice, and very few, probably, used it more than he did himself, as an amateur. His contention, however, was, that the theory of music, regarded as a science, had not been at all developed. Nothing was known, theoretically, of the different modes of dividing the octave, and he considered such a state of ignorance to be a scandal to musical science. This had led him to make experiments. He believed that now a new material for musical composition had been provided, but that was quite a secondary consideration.

Mr. ELLIS observed that Handel used the mean-tone temperament, and presented an organ to the Foundling Hospital with sixteen notes in the octave.

[1] Except, of course, from the tuner's point of view.

The CHAIRMAN said he might venture to repeat a remark he had made on the previous occasion, that if Herr Joachim played in perfect intonation, theoretically, how could he play with a piano, which was of course a perfect mass of cacophony?

Mr. ELLIS said he would, of course, alter his intonation so as to be in tune with the piano, as was shown to be the case in the experiments of Cornu and Mercadier. He had noticed the same thing himself in listening to the Paris Prize Choir of Tonic Sol-faists. They sang some pieces without accompaniment, and some with, and the intonation was as different as light and darkness.

The CHAIRMAN remarked that one of the inconveniences of perfect intonation would be, that you would continually leave off on a different keynote to that on which you began, so that the old test of sinking in pitch for singing out of tune must be entirely fallacious.

Mr. ELLIS said that Smith, in his 'Harmonics,' quoted the case of a monk who had reckoned up that the choir must have gone many commas out of tune; and argued, therefore, that there was an involuntary temperament used by singers, which he believed to be the case.

Mr. BOSANQUET said, Smith established it as a proposition,[1] that in performance by violinists or singers, there was nothing in the way of an accurate rule of any kind, but there was a constant process of adjustment, by which all difficulties were got over.

The CHAIRMAN: Art—not science.

The proceedings were concluded by a vote of thanks to Mr. Bosanquet.

[1] Smith's *Harmonics*, Prop. xxii. p. 225.