The Miracle Temperament and Decimal Keyboard

by George Secor

Introduction

It has been over twenty years since I last submitted articles on microtonality for publication. Since that time members of a new generation have discovered that there are many alternatives to the twelve-tone equal temperament, and I am encouraged to observe that interest in exploring, theorizing about, and playing music in alternative tuning systems (ATS's) has increased considerably during that time.

It is unfortunate that university music departments almost without exception strongly discourage their composition students from considering the resources that ATS's have to offer. It is not surprising, therefore, that most of the recent progress in this field has taken place outside of musical academia. This has been greatly facilitated by the creation of various forums on the Internet, such as the Yahoo Groups Tuning List <http://groups.yahoo.com/group/tuning/> and its offshoots.

During 2001 a couple of temperaments that I had written about in the 1970s were stumbled upon by members of the Tuning List, and each of these became a focal point for a series of exciting discoveries that unfolded in the course of that year. One of these was my 17-tone well temperament of 1978, which is explained in the accompanying articles by Margo Schulter and myself. The other, which is now called the Miracle temperament, I found in 1974 under very different circumstances from those in which its recent rediscovery took place.

Also during 2001 I fleshed out a keyboard design that I had sketched out over 25 years earlier, based on the generating interval of the Miracle temperament. I also started working on a new multi-system notation which I presented on the Tuning List at the beginning of 2002, and subsequent improvements on this have finally reached a point where it can be made available for use and/or evaluation by the microtonal community. (Before anyone protests, "Oh no, not yet another notation!", I think that it would be wise to look before you speak.) So it is no exaggeration to state that the past couple of years have been for me both exciting and productive ones.

To tell how all of these things came together, perhaps it would be best if I started at the very beginning.

Family Portraits

Shortly after learning how to tune a piano in my late teens, I became curious about the acoustical basis for the 12-tone equal temperament (12-ET). Upon realizing the significance of the error of the thirds and sixths of 12-ET, I decided to investigate other equal divisions of the octave and just intonation completely on my own in 1963, starting literally from scratch with a table of logarithms and a retunable electronic organ as my tools. Not many books on this subject were available, the two most valuable being J. Murray Barbour's *Tuning and Temperament* and Harry Partch's *Genesis of a Music* (first edition), and I concluded that, if I wanted to learn any more about the possibilities offered by ATS's, I was going to have to do it on my own.

In comparing the deviations of intervals from just intonation in various systems, I found that, while tables of numbers were good, a graphical method was more useful in conveying the most information in the clearest way. A single graph could show the deviations from just intonation for any number of tempered intervals as a function of the tempered fifth that served as the generator for an entire continuum of tunings. I also observed that fifths within certain ranges of size defined families of tempered systems, with the number of tones in the octave for larger members of the family being the sum of those of smaller members. If these are arranged in order of the size of the fifth, then successive members are placed between existing members to produce a sequence of numbers that may be considered a one-dimensional version of Erv Wilson's scale trees.

For each family I prepared a graph illustrating the deviations of intervals representing all of the intervals that Partch defined as consonant within a given harmonic limit, with vertical lines identifying the size of the fifth for each member of that family – a sort of family portrait. I thus identified 31-ET as the best member of the 7-19-31-12 family (for a 7 limit) and 94-ET as the best member of the 12-53-94-41-29-17 family (for a 15 limit, although 94 does rather well even beyond this).

There are instances in which tonal systems cannot be compared using the fifth as a generating interval, for example, 19-ET and 22-ET, which are in different families, due to the fact that different commas vanish in each of these divisions. To illustrate: since four tempered fifths in 19-ET are equivalent to a tempered major third plus two octaves, the syntonic or Didymus comma (80:81) is tempered out and thereby vanishes in 19-ET, while in 22-ET two tempered fourths are equivalent to a tempered harmonic seventh, so it is the septimal or Archytas comma (63:64) that vanishes.

Another instance in which tonal systems cannot be compared in this manner occurs when at least one of the systems is not defined by a single circle of fifths, such as 34-ET or 72-ET.

I eventually realized that any two divisions of the octave could be compared graphically if an appropriate generating interval could be found that is approximately the same size in both divisions. This led to my discovery of other familial relationships among tonal systems. A generator approximating a major third (~4:5) within a certain size range will define the 19-41-22 (9-limit) family, while in a different range it will define the 31-65-99-34 (5-limit) family. Other examples are the 19-72-53-87-34-15 (9-limit) family (with ~5:6 generator) and 7-31-24-41-17-10 family (for certain ratios of 3, 11, and 13 only, with ~9:11 generator). The approximate size of the generating interval for a family can be determined according to which intervals are nearest in size (but not exactly alike) in its members.

The Minimax Generator

Three systems that I considered to be among the best and most useful divisions of the octave are in the family of 31-72-41, but finding the generating interval for these was

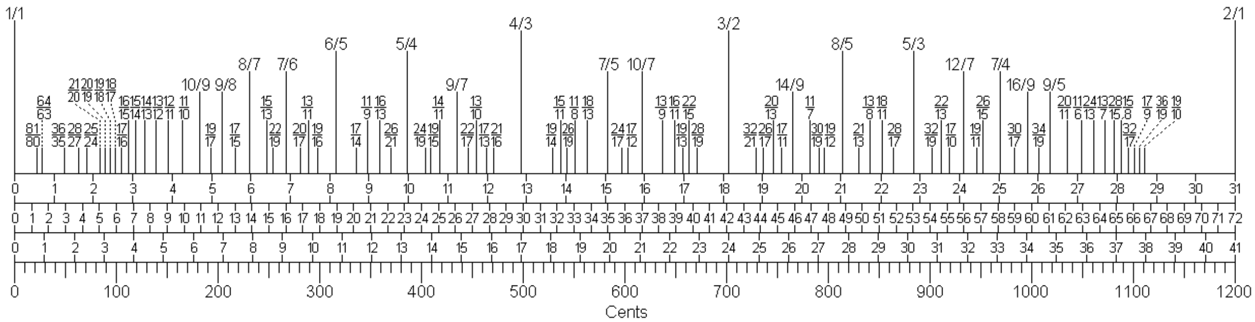

Figure 1 - The 31-72-41 Family of Equal Temperaments Compared with Just Intonation

a little more of a challenge, since it did not turn out to be an approximation of a single simple ratio. Since the approximation of 7:8 seemed to be very similar in size in all of these, I chose that interval for the x-axis in my original graph, but the tones of 31, 41, and 72-ET that most closely coincide are half of that, or approximately 1/10 of an octave (see Figure 1), which resulted in the slope of some of the lines being multiples of 1/2. This did not present any problem in plotting the lines, except that, when it came time to publish my findings, I made a error in calculating an optimal size for the generating interval and, lacking any indication of the proper value on the graph, did not realize this until later.

My original graph included all 11-limit consonances, but a simplified version is shown in Figure 2, which shows only the deviations for the tones representing the odd harmonics through 11 as a function of the generating interval. By inspection I concluded that the best value for the generator would be the one that produced the smallest range of deviations, the one that minimized the maximum error for the 11-limit consonances. Note that 7 degrees of 72-ET (7º72) comes very close to this value.

In comparing various divisions of the octave, it is not enough to determine only the deviations from prime or odd-numbered harmonics. One might conclude, for example, that major and minor triads in 22-ET are superior to those in 19-ET by evaluating only the perfect fifth and major third. It must be taken into account that such triads also contain a minor third (for which the error of the other two intervals accumulates in 22-ET while it cancels in 19-ET), so the error of the minor third should be added into the equation. The calculation becomes increasingly complicated when additional harmonic factors are considered, inasmuch as the deviation of the intervals representing each consonance within the harmonic limit (disregarding inversions) must be evaluated.

In actual practice, once the deviations get smaller than two cents, it is very difficult to distinguish an interval from one in just intonation in an actual musical performance. On the other hand, it takes only one tempered interval with a fairly large deviation to spoil the effect of a chord, regardless of how many of the other intervals have a relatively small error. Thus it is important to minimize the maximum amount by which all of the intervals within the harmonic limit deviate. This consideration also simplifies the calculation, inasmuch as it would be necessary only to determine the deviation for each odd number within the harmonic limit (including 1, which always has zero deviation) and then find the difference between the largest positive and negative deviations. In comparing the success of two different tonal systems in approximating a certain

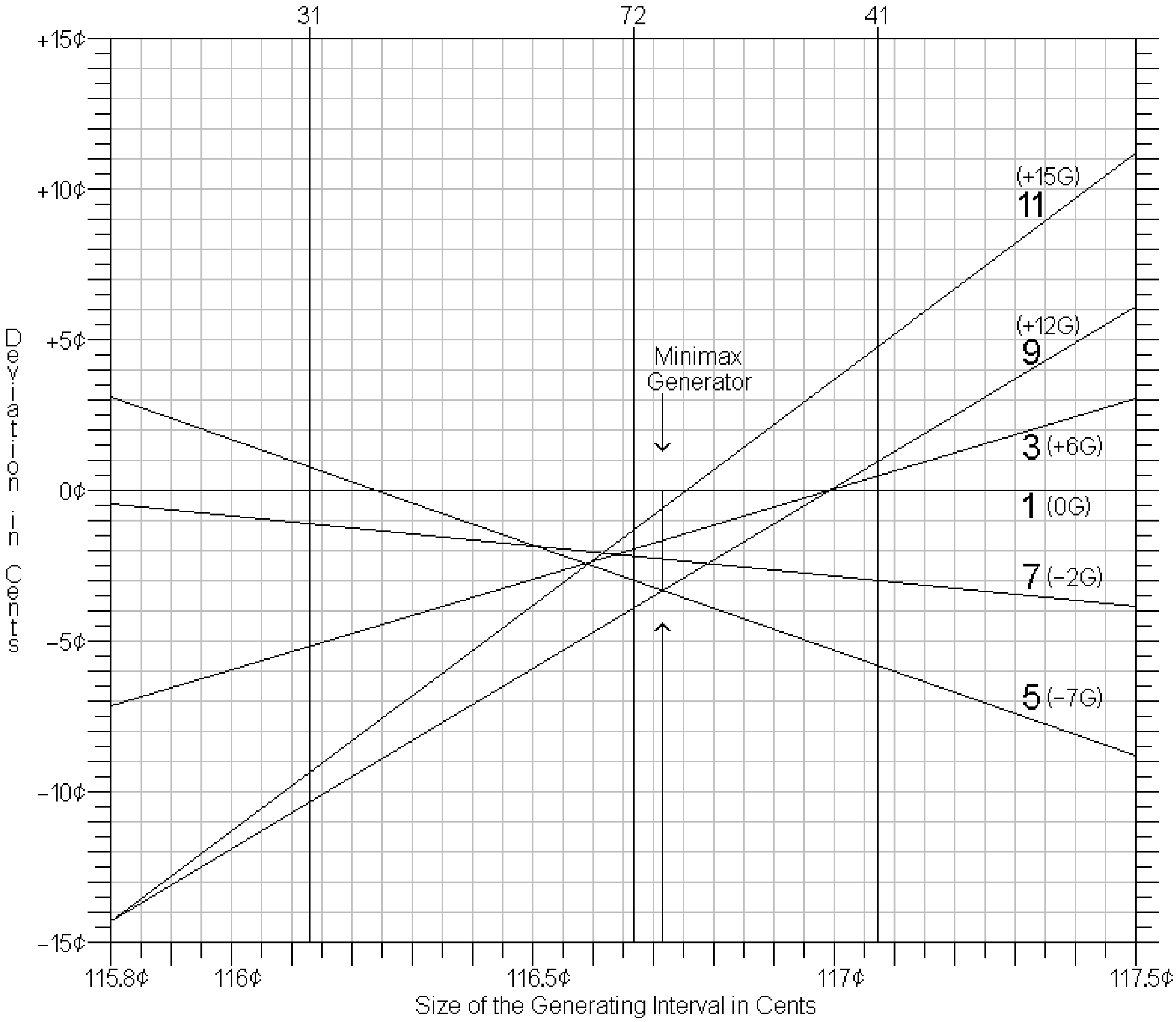

Figure 2 - Deviation of Tones Representing Odd Harmonics as a Function of the Generating Interval

harmonic limit, it is therefore necessary only to compare the *maximum difference* in deviation occurring between two odd harmonics in each system. In instances where this maximum difference can be expressed as a function of a generating interval, the optimum value for the generator is referred to as the *minimax* value, i.e., the one that minimizes the maximum error.

In the graph, the deviation of any interval representing an 11-limit consonance may be determined by finding the difference between the deviations for the two harmonics in its ratio. Notice in Figure 2 that the minimax generator for the 31-72-41 family produces exactly the same deviation for both the 5th and 9th harmonics, making the interval 5:9 exact, thus enabling the precise minimax value to be calculated (see below). This graphical method has the advantage of providing a clear picture of how each interval is affected as the size of the generator is changed. (It should be understood that the optimal minimax value is slightly different from the size of the optimal generating interval calculated using the root-mean-square method.[1])

In order to construct the graph, it was necessary to locate the proper position in the series of tones thus generated for each of the harmonics in order to determine the slope of the line illustrating its deviation (labeled in the graph as coefficients of the generator, G). In the process I discovered that a sequence of 23 tones separated by this generator was sufficient to produce not only an 11-limit hexad (6:7:8:9:10:11) but also ten 7-limit tetrads (4:5:6:7). Each additional tone added to this sequence would then result in both an additional hexad and tetrad, making for an impressive economy of tonal resources, to say the least.

I quickly realized that a keyboard could be based on this generating interval that would accommodate 31, 41, and 72-ET, as well as a temperament based on the 11-limit minimax generator. The best feature was that, for all of these tunings, the economy of resources translated into a very compact and efficient tonal mapping, so that 11-limit harmony would be playable on this keyboard without having to reach excessively far along a front-to-back direction (or y-axis).

A Tale of Two Keyboards

I discovered all of this before I learned of the xenharmonic movement in December 1974. Earlier that year I had attended a demonstration of the Scalatron (digitally retunable electronic organ) prototype, and recognizing that conventional keyboards were not the best way to perform music with more than 12 tones in the octave, I unwittingly proceeded to re-invent the Bosanquet generalized keyboard and subsequently approached the Motorola Scalatron company with the proposal of employing it on their instrument. (While I had previously seen a diagram of the Bosanquet keyboard in Helmholtz's *Sensations of Tone* some years earlier, the shape of Bosanquet's keys was so different from what I was considering that it didn't even occur to me that the underlying tonal geometry might be the same.)

Around that time several members of the xenharmonic movement had gotten in touch with Scalatron president Richard Harasek and sent him copies of the first two issues of *Xenharmonikôn*, which he passed on to me and which I promptly read. The second issue included Erv Wilson's diagrams of a modification of Bosanquet's keyboard, with hexagonal keys, at which point it became clear that my keyboard proposal was not new. The important thing was that I now had good confirmation that I was on the right track.

When John Chalmers and Erv Wilson learned of my microtonal work, they invited me to submit something for the next issue. Harry Partch had passed away shortly before *Xenharmonikôn 2* was released, and John Chalmers put a brief message at the end of his *Notes and Comments* column: "For *XH3*, may I suggest that we submit material either inspired by Partch's lifework or built upon it? The use and development of the concepts he pioneered is a more fitting memorial than any eulogy could ever be."

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[1] The RMS 11-limit optimal values are 116.6723 cents (unweighted) and 116.6776 cents (weighted, treating 1:3 and 3:9 separately), as compared to the minimax value of 116.7156 cents.

With the deadline for submissions fast approaching, I hastily put together a short article that included a sketch for my other new keyboard geometry that was based on the generating interval of approximately 1/10 of an octave, which could be used not only for 31, 41, and 72-ET, but also for Partch's 43-tone just intonation, inasmuch as the latter could be mapped uniquely into a 72-tone octave. I also mentioned the low-error (near-just) 11-limit regular temperament that was also possible on this keyboard and that the same fingering patterns were maintained across all five tonal systems. This "decimal keyboard" (as I have recently named it) was thus presented as an alternative to the Bosanquet generalized keyboard for the 31 and 41 divisions; it would also easily accommodate the 72 division, something that was not possible with the Bosanquet keyboard.

For the remainder of the year I was heavily involved in the generalized (Bosanquet) keyboard Scalatron project and, after that, in using it to explore new tunings. In effect, the keyboard that I had discovered was destined to be overshadowed by the one that I had rediscovered. In the years following, I gave very little thought to the decimal keyboard or to its underlying tonal geometry, and as far as I can tell, so did everyone else for the rest of the 20th century.

The Miracle Temperament

I first learned of the current interest in my *Xenharmonikôn* article in the course of my recent e-mail correspondence with Margo Schulter:

> *MS (9/25/2001): There was a lot of excitement this Spring when some people on the Alternate Tuning List came up with a near-JI tuning using a generator of around 7/72 octave -- and someone then discovered your article in* Xenharmonikôn 3 *(Spring 1975), "A New Look at the Partch Monophonic Fabric," describing an identical keyboard scheme with a generator of 116.69 cents.*
>
> GS (9/28/2001): You might want to pass the word along that I made a mistake in the calculation and issued a correction in *Xenharmonikôn 5* (on page 2 of my first article in that issue, Spring 1976): the correct value was to be an interval that, when carried out to nineteen places, would generate a just 5:18 (i.e., 5:9 increased by an octave). I gave the revised value in cents as "(1017.596288 + 1200)/19 = 116.7155941. This makes the intervals 8:9 and 4:5 false by 3.323 cents; altering the value slightly will improve one and worsen the other. No other primary interval in Partch's 11-limit [i.e., 11-limit consonance] has a greater error, making this the optimum value for the generating interval. …"
>
> *MS: You might be amused to know that people have coined the term "secor" (lowercase) for a generator of around this size. For example, a 21-note tuning of this type is said to "span 20 secors."*
>
> GS: That has to be a first. As far as I knew, the only intervals named after anyone, living or dead, were commas, and those honored (i.e., Pythagoras and Didymus) had to be dead for a very long time to qualify. … Perhaps by dropping out of sight and lying low for so many years I have given the impression of being as good as dead, so that someone thought it fitting to bestow the honor.

As I later learned, there has been a lot of activity in the microtonal community over the past couple of decades involving the exploration of new scale systems and the identification of various commas and schismas, some of which would inevitably receive names associated with their discoverers. So while this wasn't a first, at least it was gratifying that, just as I had rediscovered Bosanquet's generalized keyboard geometry in 1974, so had others rediscovered my 1974 decimal geometry some 27 years later.

A few days after this I received a very brief letter from Erv Wilson, with one of the attachments showing my keyboard geometry in a hexagonal key layout for the Partch 43-tone just intonation. I replied with a counter-proposal using rectangular keys in a 6-color layout that I had devised earlier in the year for inclusion in a book which I am writing (see Figure 3). His response was quite enthusiastic, and I decided that I should write an article about it for *Xenharmonikôn 18*.

Since I had just begun working on another article about the 17-tone revolution in which Margo Schulter and I were involved, I didn't visit the Tuning List website to investigate any of this until after I had finished the article in mid-December. In the meantime Joseph Monzo invited me to join the Tuning List and also referred me to his online tuning dictionary (now located at <http://sonic-arts.org/dict/>), where I saw that my temperament had been given the acronym "MIRACLE" (**M**ultitudes of **I**nteger **R**atios **A**pproximated **C**onsistently, **L**inearly, and **E**venly), with the minimax generator identified as the "secor". A number of moment-of-symmetry (MOS) scales are also mentioned in connection with this generator (or its 72-ET approximation), including a couple with the colorful names "Blackjack" (21-tone) and "Canasta" (31-tone) after card games in which those numbers have significance. (The 41-tone MOS scale was later given the name "StudLoco" after a Mexican version of poker.) My *XH3* article did not mention any MOS scales for a couple of reasons: 1) This was a new term that first appeared in print in Erv Wilson's articles in that very same issue; and 2) having little interest in any near-just tuning less than 13-limit, I did not attempt to determine the possible scale resources.

A brief summary of the events that culminated in the rediscovery of the Miracle temperament by members of the Tuning List was given in David Keenan's reply to my message on the Tuning List congratulating him and Paul Erlich for their accomplishment:

> … we stumbled upon Miracle while attempting to answer [Joseph Pehrson's] question, which was "What is the best 19 note subset of 72-tET". We understood "best" to mean maximising the availability of near-JI harmony while also having reasonable melodic properties. There seemed to be several options with no clear winner. I pointed out that most were coming from a certain 31 note subset (which we now call Canasta) and Paul pointed out that this was a MOS with a generator of 7/72 octave (which we now call a … secor) and that this generator also had a 21 note MOS (Blackjack).[2]

Paul Erlich's description of the "true nature of the Blackjack scale" is rather technical. According to the terminology described in his paper, *The Forms of Tonality*[3], the

[2] David C. Keenan, message #32782 of January 15, 2002, posted to <http://groups.yahoo.com/group/tuning/message/32782>

[3] See <http://lumma.org/tuning/erlich/erlich-tFoT.pdf> for a detailed explanation of how unison vectors may be used to define certain scales as periodicity blocks.

diatonic scale can be derived from the 5-limit lattice by positing 81:80 as the *commatic unison vector* and 25:24 as a *chromatic unison vector*. This specification thereby determines its melodic, harmonic, and notational properties. The Blackjack scale is not qualitatively different, since it can be derived from the 7-limit lattice using 2401:2400 and 225:224 as commatic unison vectors (more are needed for 11-limit, fewer for 5-limit) and 36:35 as a chromatic unison vector.[4]

One thing that may come as a bit of a surprise is that in all of the years since I wrote about the Miracle temperament, I have never tried it, so I didn't even hear it until I listened to Joseph Pehrson's “Blackjack” composition for trombone and recorded electronics. I never had any use for the Miracle temperament because it doesn't have ratios of 13 (and, lacking 13, the 8:9:10:11:12:14 hexad is not a constant structure), so I wasn't particularly interested in a near-just 11-limit tuning. And it never occurred to me to use scales such as Decimal (the 10-tone MOS) or Blackjack for composing music. For this reason I regard its rediscovery as an accomplishment that is more significant than mine, the outcome of a deliberate effort that has resulted in some actual music.

Of more value to me was the keyboard geometry that is derived from the Miracle temperament, which brings us to our next topic.

The Decimal Keyboard

This keyboard is a departure from the elongated hexagons and ellipses that Erv Wilson and I used for the Bosanquet generalized arrangement. I initially tried a hexagonal shape, but the sort of hexagon that would be indicated by the keyboard geometry would have one pair of sides so short as to be almost rectangular, so a rectangular shape would be preferable for its simplicity.

The keys in Figure 3 resemble computer keys, but they are actually narrower and taller – narrower for an better octave reach, and taller to make it easier to avoid the accidental pressing of two keys in adjacent rows by the thumb. I don’t believe that the shorter lateral distance would be a significant problem, since the side of the thumb would generally be parallel to the long dimension or, when stretching, the diagonal dimension of a key.

The “decimal keyboard” concept is illustrated by referring to the generating interval scale at the left of the figure, in which the center of the leftmost “C” key is aligned with the zero-generators (0G) mark. The keys immediately above C are aligned with the marks that are multiples of ten: +10G, +20G, etc. The keys to the right of C have their centers aligned one mark higher in succession, making the key ten places to the right the octave of the key directly above the starting point; in the Miracle temperament this key would play the tone 10 secors above C.

In a computer interface, the generator number could be used for easy integer identification of each pitch (within an octave), with the ten’s digit being the number of keys along the y-axis (including the sign of the integer) and the unit’s digit being the number of keys along the x-axis, C (or 1/1) being the origin; units would thus range from

[4] Paul Erlich, message #24392 of June 5, 2001, posted to <http://groups.yahoo.com/group/tuning/message/24392>

0 to 9. Such integers could be stored digitally in a single byte of information in the range ±127, with -128 serving to indicate a rest.

The color coding is based on the six circles of fifths in 72-ET. I initially tried six colors (one for each circle of 12 keys), with light (for 7 keys) and dark versions (for 5 keys) of each color (e.g., white and black, pink and red, etc.), but the resulting pattern is so complicated as to be almost impossible to comprehend.

By trying various modifications of this initial idea using fewer colors, I finally arrived at one based on the two sets of 36-ET in 72-ET, retaining the 7 vs. 5 contrast within each circle of fifths. In 72-ET neighboring keys in the same column differ by 2º72, so that 3 contiguous keys in a single column belong to the same set of 36. If these are all colored using the same 7 plus 5 pattern in all three circles, the sets of 5 (dark) in each circle can be readily identified on the layout in sub-groups of 2 and 3, just as on the Bosanquet keyboard. These are the black and red keys on the decimal keyboard.

For the light colors to be paired with black and red, I chose their complementary colors: white and a light cyan. Inasmuch as this did not result in easily distinguishable sub-groups of 3 and 4, I hit upon the idea of making the outer keys in each column set of 3 a grayish version of the central key, i.e., gray and grayish-cyan. This then makes the 7 white and 7 light-cyan keys easy to identify in sub-groups of 3 and 4. Since the white keys are the naturals C through B and the light-cyan keys are the seven tones 1º72 lower, the tones in a so-called “just major scale” on C are played entirely on white and light-cyan keys. (A full-color version of Figure 3 may be viewed at <http://groups.yahoo.com/group/tuning-math/files/secor/kbds/KbDec72.gif>.)

In 41-ET, these corresponding tones in the white and light-cyan groups are a single degree (or comma) apart, whereas in 31-ET they are duplicate keys for the same tones. These relationships apply to the black and red groups and the gray and grayish-cyan keys as well.

In Figure 3 the keys are labeled with 19-limit ratios according to a 72-tone mapping. Since there are more than 72 keys per octave, it is possible to assign more than 72 pitches per octave on the decimal keyboard so that like intervals will still occur in like patterns, with keys that would be duplicates in 72-ET being assigned ratios that differ slightly in pitch. For example, 14/13 is at +32G, while 13/12 is at -40G.

Figure 3 shows a keyboard with 101 keys per octave, which is the minimum number that I would recommend for 72-ET, but a system of fewer tones, such as 41-ET or a MOS of the Miracle Temperament, would require considerably fewer keys.

The rows of the keyboard are right-rising rather than left-rising, for a very good reason: This puts the third of all just major triads in a position where it is more easily played by the thumb, just as occurs with positive tunings on the (right-rising) Bosanquet generalized keyboard. The advantage of this is immediately evident for the right-hand first inversion and left-hand second inversion major triads (in closed position), and especially for the open root position (spanning a tenth) for the left hand. For the closed root position triad for either hand it is a simple matter to play the root and fifth with the index and little fingers and cross the thumb under to play the third. The subminor

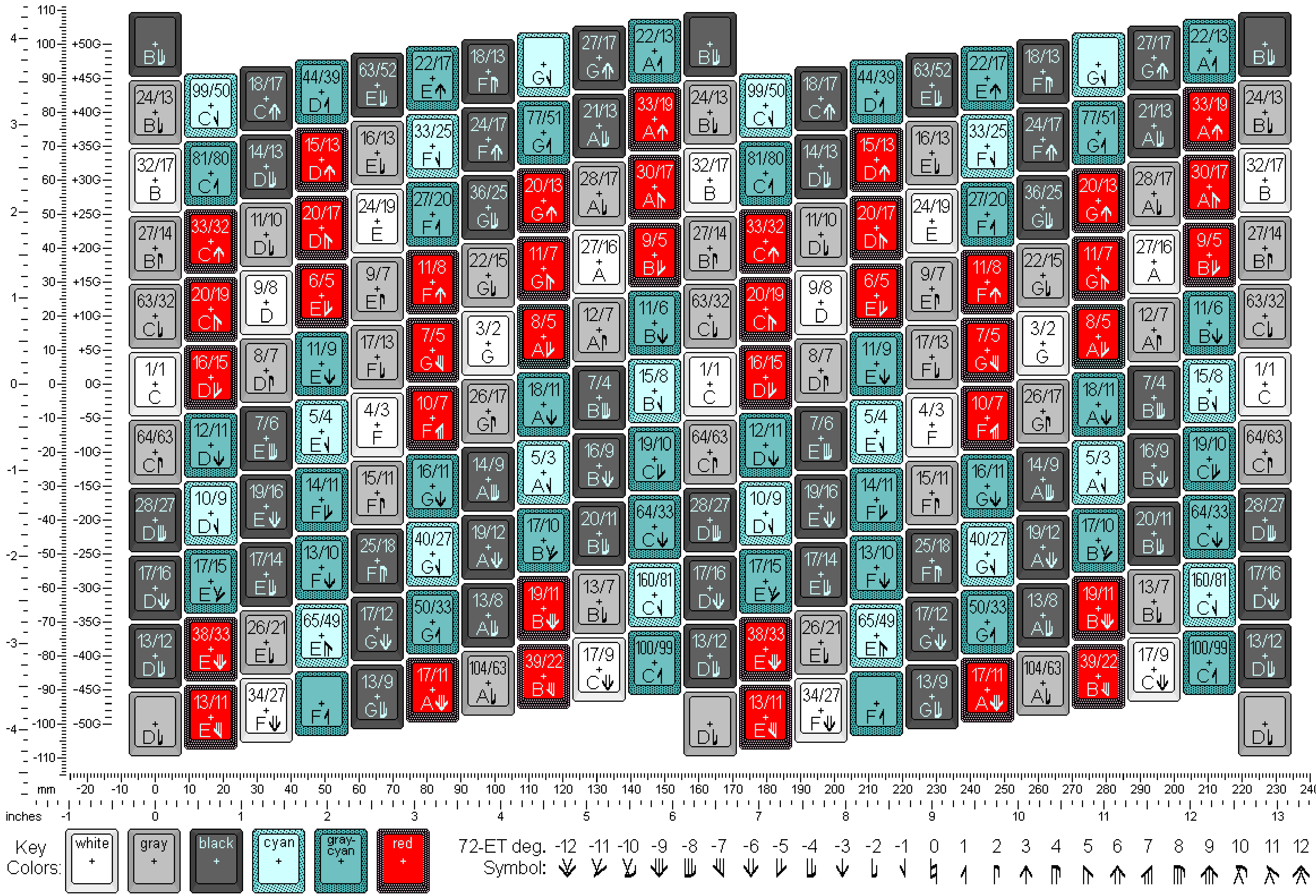

Figure 3 - The Secor Decimal Keyboard in 6 Colors

(6:7:9) triad is played similarly. In the case of the minor triad the playing technique would be slightly more difficult, so for tunings with a finite number of tones ("closed" systems) it would be advisable to have an adequate number of duplicate keys in order to simplify the fingering.

The Sagittal Notation

In addition to the ratios, the keys in Figure 3 are also labeled with a new notation for 72-ET, which requires some explanation.

I began development of this notation in August 2001. Shortly after joining the Yahoo groups Tuning List at the end of that year, I presented what I had developed and offered to consider suggestions for improvements.

But I did not even begin to imagine how much it could be improved! Many people offered helpful suggestions, and for the next year-and-a-half David Keenan and I worked together to expand my original idea into a notation that would be both versatile and powerful, but for which the required complexity would not make it more difficult to do the simpler things. An introduction to the resulting "Sagittal" system of notation is presented in the accompanying article.

For the latest information regarding details of the notation and related resources, please visit the Sagittal website.[5] Users of the Scala tuning software may also download the latest version, which includes a partial implementation of the Sagittal notation.[6]

Conclusion

Much has occurred during the twenty-odd years that I have been away from the microtonal scene. Since my return I have made many new friends on the Yahoo Tuning List and have had a chance to participate in a highly stimulating exchange of ideas and opinions. That forum has made it possible not only to keep up with the latest events, but also to establish, clarify, and accomplish objectives in collaboration with others much more efficiently than would have been possible years ago, if at all.

I am also delighted that *Xenharmonikôn* is still alive, well, and flourishing as a different sort of forum, one in which we can share the accomplishment of our objectives, and I wish to extend my gratitude to those who have kept it going for so many years.

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[5] Located at <http://users.bigpond.net.au/d.keenan/sagittal/>, the Sagittal website will offer an introduction to the notation (consisting of a tutorial, along with the remarkable story of how Sagittal came to be), a TrueType font designed for use with commercial musical notation software, and links to other resources.
[6] Scala is freeware produced by Manuel Op de Coul and is available at the Scala Home Page, located at <http://www.xs4all.nl/~huygensf/scala/>