ACOUSTIC METHODS OF WORK

IN RELATION TO

SYSTEMATIC COMPARATIVE MUSICOLOGY

INCLUDING SOME ACOUSTIC TABLES

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CHAPTER IV

THE AUTHORITATIVE TONE-SYSTEM, THE GOLDEN SYSTEM AS AN OBJECTIVE MEANS OF VALUATION.

However, in order to pass judgement on the various historical tone-systems, the chief defect of which is the displacement of **Commas** between for example »d and d+«, »des — and des« etc., it is necessary first to construct an authoritative tone-system **without** these Comma-displacements, which claims that many tones with plus would be made much lower, and many tones with minus much higher.

In August 1930 I found the new Fifth in two ways which gave the same result.

Group A. Arithmetic Series I—VI, retrograde and direct.

The sum of 2 golden adjacent Nos. forms the following No., and the sum of the **cents** of two golden adjacent tones forms the following cents in the same golden Series.

1. The astronomer J. Kepler (1571—1630) combined tones with the sum of the numerators and the sum of the denominators respectively of 5 tone-fractions:

$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$ Partial-tone 840 5 cents
C	c	g	a	as	gisis
					The golden gisis........... 843.2 »

In the 19-toned and the 31-toned golden section respectively we **transform** the tone-sequence in this manner (see **Scheme 10**):

							Double super division		
Fourth-Series I retrograde:	19 toned	.....	.....	1 cis	2 des	3 d	5 es	8 **f**	13 as
	31 —	.....	1 deses	2 cis	3 des	5 d	8 es	13 f	21 as
	50 —	1 bisis	2 deses	3 cis	5 des	8 d	13 es	21 f	34 as ↓

→

Ludwig Sonnenberg, principal teacher in Bonn (1820–88) called the sequence 1, 2, 3, 5, 8, 13: »Kepler's Series« (Notes 12–13), our golden **Fourth**-Series I retrograde, in 1844 [illegible] to No. 17 by Gabriel Lamé, in 1929 up to No. 40 by L. Kaiser.

2. Dr. [illegible] Kaiser, the mathematician, found by pure mathematics (Note 14) (without reference to tones) a similar sequence; we call the latter »Kaiser's Series«, our **Fifth**-Series II, which we use in the following manner:

Fifth-Series II retrograde			Double super division.			
	19-toned	4 dis	7 fes	11 g	18 ces	
	31- —	7 dis	11 fes	18 g	29 ces	

3. In May 1930 Andreas Kornerup, engineer, directed my attention to the fact that the Fifth in these systems has tone-Nos. which form similar sequences of Nos. for example $\frac{\text{Fifth}}{\text{Octave}} = \frac{7}{12}, \frac{11}{19}, \frac{18}{31}, \frac{29}{50}, \frac{47}{81}$ etc.

We name the sequence 12, 19, 31, 50, 81 the **Octave**-Series III, Andreas Kornerup's Series, the denominators of the fractions.

a) It occurred to me to calculate these fractions in cents, which I did, August 15th 1930. The value moved like a **pendulum** quickly approaching the point of balance, where the seventh decimal would be stable at the Fifth $\frac{1200 \times 3571}{6155} = 696.2145$ cents (Acoustic statics).

Fourth $\frac{1200 \times 2584}{6155} = 503.7855$ error 3571

The temperaments, the octaves of which are the denominators of these fractions, I called the **organic** temperaments, and, at the same time, it occurred to me to construct an **authoritative** »tone-system of the Fifth« with **this** Fifth which, however, **also** generally appears by means of the super-division (golden cut) of the octave, **Scheme 10**, Series III. I calculated the Fifth on August 17th 1930, as shown below:

b) The fraction of super-division	.... Cents	.... Series III:
ω	0,618.03398 × 1200 gives 741.64078	.... direct ases
$1 - \omega$	0,381.96602 × 1200 ..... 458.35922	...... retrograde eis
Total 1,0	.... 1200 cents	.... c'

Further { ases is 11 Fourths from c | and 4 octaves back then f = 503.7855
eis is 11 Fifths — - | — 6 — — — g = 696.2145

Total = 1200 cents

In 19-toned and 31-toned golden sections respectively we use »Andreas Kornerup's Series« in the following manner:

Octave-Series III retrograde			Double super division		
	19-toned	O c	7 eis	12 ases	19 c'
	31-toned	O c	12 eis	19 ases	31 c'

The difference between golden eis = 458.4
and syntonic — = 457.0

Schemes 14 and 15: nearly k = 1/15 Comma = 1.4

4. Out of the calculated golden system I have formed some other Series for instance 31-toned see **Scheme 20**:

			Double super division.			
Great Third-Series IV:	4 cisis	6 eses	10 **e**	16 ges	26 bes	
Tritonos — V:	....		9 feses	15 **fis**	24 beses	
Great Sixth- — VI:	....		9 disis	14 geses	23 **a**	

Group B. Geometrical Constructions of golden Tones.

The golden tones and intervals can also be constructed geometrically as shown in **Scheme 8**, which indicates:

1) above, part of a regular **Pentagon**, the angle of which $= 108^{0}$ is divided into 3 parts by means of 2 chords; if the chord be equal to a Tetrachord — the interval C-F, (the Fourth f), this interval will be super-divided in »**great** double super-division«

in { C ——— D ...... Es ——— F = Series I
192.43 + 118.93 + 192.43 = 503.79 cents }

2) below: a **square** placed inside a semicircle, whereby the diameter is super-divided into a »**small** double super division«

in cents {
Cis .... D ——— E ..... F
D ..... Es ——— F ..... Ges
B ..... C ——— D ..... Es
118.93 + 192.43 + 118.93 = 430.29

corresponding with: C ..... D ——— F ..... G
192.4 + 311.4 + 192.4 = 696.2

3) further: **within** the square we can form a »**great** double super division«

in { D —— Dis .... Es —— E
73.50 + 45.43 + 73.50 = 192.43 cents }

Scheme 9 shows the golden system constructed by means of the **Pentagram** with the same chord C .. F, so that the side is = C .. Es in the Pentagon.

Fourth-Series I: C, Deses, Cis, Des, D, Es, **F** and further As, Des'.

Further: $\frac{\text{C..(Es)}}{\text{C..E}} = \frac{\text{es}}{\text{e}}$ = Cosinus 36^{0}, the relation between small and great Third.

Scheme 10 shows »double super-division« in 4 Series; the upper and lower edges of a square are divided:

{ from c upwards into ases, direct,
— c' downwards into eis, retrogade.

If we continue a single super-division,

1) direct, upwards, we will get **decreasing** intervals upwards,

2) retrograde, downwards, we will get decreasing intervals downwards. We use oblique lines { from the corner upwards on the right / — — — downwards — left } forming 8 double super-divisions, which give the result:

	upwards, retrograde:	Cents	downwards, direct, Scheme 10:
↑ Series I	c—es.. **f** —as	815	e—g .. a —c' ↓
— IV	c—**e** .. ges—bes	1008	cis—f .. gis—c'
— II	c—fes .. **g** —ces	1126	d—fis .. as—c'
— III	c—eis ——— **c'**	1200	c ——— ases—c'

We continue double super-division only in Series I:

Series I:	c—des .. d —es	311	a —bes .. b —c'
	c—cis .. des—d	192	bes—b .. ces—c'
	c—deses .. cis —des	119	b —ces .. bis—c'

The tones are in pairs supplement-tones (making an octave together) for instance:

Series I	retrograde	deses	cis	des	d	es	**f**	as	c'
	direct	bis	ces	b	bes	a	g	e	c

Scheme 20 shows golden tones in 6 Series formed geometrically by means of parallel lines in a Pentagon, for instance:

The tones :	Eis	F	Fis	Ges	G	As	A	C'	Des'
Series I-III:	III.	I.	....	....	II.	I.	....	III.	I.
— IV-VI:	....	....	V.	IV.	....	....	VI.	....	....

Scheme 21 shows cents (to 4 decimals) for an organic 19-toned section of the infinite golden tone-system with two Molecules:

{ t = 73.501 cis, which is the smallest interval in this section,
v = 45.426 deses, the greatest interval in the next **organic** section on 31 golden tones.

	v.		t.		cents
The Molecules in 12-toned golden section	5 × cis	+	7 × des	=	1200
19- — — —	7 × deses	+	12 × cis	=	1200
31- — — —	12 × bisis	+	19 × deses	=	1200

v. outside the section.

Group C. All Comma-displacements disappear.

Example: $d = \frac{10}{9}$ and $d+ = \frac{9}{8}$

merges into the golden tone d = 192.43 cents, approximating to the **secondary** halving of the string between d and d +, which in the Formula X gives:

	180		
	162	161	160
	d	....	d+
Particles	648	644.3	640
cents	182.4	193.1	203.9

or $\frac{180}{161} = 193.12$ cents, the difference is inaudible,

Why not $\frac{19}{17}$ = 192.56 cents? AH 8/20/78

so that the golden tone can easily be found approximately on the string.

The **tertiary** halving gives a somewhat higher tone (Formula Y):

$$\frac{160 \quad 161 \quad 162}{144} \text{ or } \frac{161}{144} = 193.20 \text{ cents.}$$

The structure of the triple chord is often the golden cut directly put to use (x = great Third, y = small Third, (2) and (3) formations):

Triple chord:	Symbol	Intervals	Small Sixth with golden cut:	
Major 2nd form	xy (2)	e g c	retrograde: 311 + 504 =	815 cents
Minor 3rd —	yx (3)	g c es	direct: 504 + 311 =	

For teachers of harmony the golden system can thus also be of use through its clear and logical construction; thus xyy (1) the Dominant quadruple chord c e g bes will, on the 1st step, be **resolved** into xy (3) = the tonic triple chord on the 3rd step: (c) f a c' (on the Tonic f) by which means the tones e, g and bes **glide** either 119 **or** 192 cents, »c des« and »c d« respectively, systematically up **or** down, with the intervals in cents:

From	o	385		696		1007	1200.	Symbols
	c	e		g		bes	c'	xyy (1)
Gliding in cents		→			←	→		
		119	←	→	119			
			192	192		192		
to	c		f		a		c'	xy (3)
	o		504		888		1200	

Since all Comma displacements disappear in the golden system, one can therefore choose any scale whatever on any golden tone as Tonic (key-note) with the smallest **possible number** of tones; the authoritative Golden tone-system represents in this field the formula for the principle of **the smallest activity**, Nature's **economic minimum** principle. It is the system with the smallest possible number of tones (acoustic Okology), an outcome of Nature's wonderful **power of adaptation**, — the supporting principle of all life in Nature.

Andreas Kornerup has called the fraction of the golden cut 0,618.034 Omega, which is recommended as an international expression in the formula $\frac{1}{\omega} = \frac{\omega}{1 - \omega}$ or: $\omega^2 + \omega = 1$ or: eis + ases = c', Series I, or 458 + 742 = 1200 cents, which we here designate, once and for all, as the authoratitive principle of relativity in the field of acoustics: The essential formula for **the universal sense of harmony.**

Even if the composers do not know of this formula, the Danish Physicist H. C. Ørsted (1771—1851) is indeed right in saying: »The work of the composer is based on mathematics although in a **deeper** measure than has ever dawned upon us.« (Note 15). It will be the task of the future student of the theory of music to get to the bottom of this deeper-lying **law** of Nature, the golden cut, the super division — the basis of the future renaissance of harmonics.

CHAPTER V

TEMPERAMENTS.

Tone-systems with 1 Molecule only.

The Octave-Series III, Andreas Kornerup's Series, marks the boundary of the **organic** temperaments, i. e. the only **rational**, the only temperaments fit for use, namely the sequence: 12, 19, 31, 50, 81 — with 19-toned as the **practical** and with 31-toned temperament as the **Standard**-temperament, see **Schemes 12** and **18**.

How many tones will be required for pianos, organs etc. is indeed a question; but, for practical measures only the organic **19** and **31** are efficient; all the others are unworkable. This is the authoritative judgement passed on the matter in question.

›Ceterum censeo: If we wish to abandon the 12-toned pianoforte we can in no circumstances choos any other than the practical 19-tonic temperament, and for finer requirements the Standard 31- (or the 50-) toned temperament.

All the inorganic temperaments ought to be excluded.

Scheme 11 shows, by way of example, how the Fourth-Series I, Kepler's Series, is carried through logically, in 19- or 31-toned temperament, where the sum of cents for two neighbour tones gives the next tone in the Series, but is split in, for instance, the 24-tone system, which is therefore authoritatively considered to be unworkable.

Further: Dr. P. S. Wedell and N. P. J. Bertelsen, actuary, Copenhagen, in January 1915 proved, by means of »the method of the smallest squares«, that the 19-toned is better than the 12-toned temperament, and that the 31-toned, again, is better than the 19-toned. To have this judgement expressed in numbers it can easily be calculated how much tne single tone in the different temperaments deviate from the golden tones, and by summing-up such deviations for 35 tones (the tones of the 7 white keys and ♯, ♭, 2♯ and 2♭) the result is found to be as follows see **Scheme 12**:

Deviation for	12-toned	Temperament	1174	cents	estimated at	100 %
	19- —	—	458	—	thus	39 —
	31- —	—	174	—	—	15 —
	50- —	—	67	—	—	6 —

In comparison it may be quoted that in the pure consistent Pythagorean system, the collective deviation for the same 35 tones is 1780 cents or 52 % **greater** than for the 12-toned temperament.

Among the numerous observations as to the change of the piano-temperaments we shall quote the following few selections:

Director Gotfred Skjerne says (1909): »We are indebted to the tempered tuning for [illegible]ous musical progress ..... but the ear is, as a matter of fact, **coarsened.**«

[illegible]or Dr. José Würschmidt showed 1920—28 »that a division of the octave into 18, 24 or 36 parts can represent **no** natural extension of our tone system, but that we have before us such an extension in a division of **19** steps. (Note 17).

Professor Joseph Yasser, New York, recommends the 19-toned temperament with the purpose: »to **enrich** our musical language, particularly when one takes into account the **growing** significance of the independent twelve-tone foundation in modern music«. (Note 18).

Professor Louis Kelterborn, Neuchâtel, (Note 19) proposes, further, on practical grounds, that simultaneously with the introduction of the 19-toned temperament, the pitch of »the tone a« should be made **a little lower**, so that **c** may keep the same pitch as it has now.

Scheme 13 shows, geometrically, an equilateral hyperbole through the tones No. 5 in various organic temperaments forming the tones in the Golden Fourth-Series I, retrograde, (with approximate pitches of tone).

Scheme 18 shows lines through corresponding tones in 5 temperaments.

CHAPTER VI

CHANGE OF THE SYNTONIC SYSTEM INTO GOLDEN TONES.

a) In **Schemes 14 and 15** we have the syntonic system, built on Pythagorean Fifths (the Partial-tone = 3/2) in horizontal lines and the Thirds = 5/4 in diagonal lines with an angle of 60°, according to the proposal of the Japanese Shohé Tanaka in 1890.

The axis through »feses, c or gisis« in the schemes is called **Zero**-Axis, because the difference between the syntonic and golden feses (equalling **15** golden Fourths) is almost nil, just as is the difference between the corresponding supplement-tones, syntonic and golden gisis (equal to **15** golden Fifths), which is seen in the following table:

4 syntonic Thirds.......... ......give	1545.255	cents
minus 1 — Fifth, the Partial-tone 3/2 = g	701.955	—
gives syntonic **gisis**	843.300	—
15 golden Fifths minus 8 Octaves, golden **gisis**, is	843.217	—
inaudible difference	0.083	—
Further: syntonic Third, the tone 5/4 = e	386.314	—
4 golden Fifths minus 2 Octaves: golden e.....	384.858	—
Slight deviation	1.456	—
1/15 of a syntonic **Comma** 21.5062 cents is k =	1.434	—
inaudible deviation	0.022	—

All the tones in **Scheme 14** can thus, in a practical way, be changed to golden tones by adding or subtracting a number of »k«. These numbers are regularly grouped in all the lines which can be enclosed through the tones in the Scheme, i. e. with regular rise in numbers of k, so that the Scheme will be reminiscent of mathematical number-designs, or **number-figures,** constructed by the Danish astronomer Thorvald Nicolai Thiele (1838—1910) in 1872 (Note 20).

Scheme 14 is thus divided into 9 symmetrical **figures**, each comprising 15 tones, formed on about **9** central points of, respectively + 15 or 0 or — 15 k:

	+	Zero	—
	gisis — + 15	gisis 0 0	gisis + — 15
	c — + 15	c 0	c + — 15
	feses — + 15	feses 0	feses + — 15
resp.	from + 22 to + 8	from + 7 to — 7	from — 8 to — 22

Thus, outside the Zero-Axis the tones are here changed by »addition on the left side«, »subtraction on the right side« of fifteenth parts of a Comma, increasing evenly in all directions.

Axis:	Direction:				Result: Central-points
1) Pythagorean Fifth-Axis, horizontal ... then one Small Third-step	to the right downward	o. c	— 4. g	— 8. d + ...	— 12. a + — 3. — 15. c +
2) Pythagorean Fifth-Axis, horizontal ... then one Great Sixth-step	to the left upward	o. c	— 4. f	+ 8. bes —	+ 12. es — + 3. + 15. c —
3) Great Third, Axis, 60° upward ...... then one Great Sixth-step	to the right — — left	o. c	— 1. e	— 2. gis	— 3 bis + 3 0 gisis
4) Small Sixth-Axis 60° downward ..... then one Small-Third-step...........	to the left — — right	o. c	+ 1. as	+ 2. fes	+ 3 deses — 3 0 feses
5) Small Third-Axis **through** the figure .	from »+ 15 gisis —« (through »o. c«) to »— 15 feses +«.				

If all the Third-lines, inclining 60°, be elongated they will cut the Zero-Axis in nothing but Zero-points, — ad infinitum.

Two examples of the division of 1 Comma will finally be quoted:

d +	sunk	8 k = 11.4700	
d	raised	7 k = 10.0362	
	Total	15 k = 21.5062	

	es	sunk	3 k = 4.3 cents
	es —	raised	12 k = 17.2 —
5 × 3 k =	Total	15 k = 21.5	— = 1 Comma.

The golden d = 192.43 cents is only a trifle below the **secondary** halving of the distance on the string between d and d +, as above mentioned (Chapter IV, Group C).

Analogously the distance on the string between Pythagorean es — $= \frac{32}{27}$ and the syntonic es $= \frac{6}{5}$ is secondarily **5**-parted according to the Formula X:

	480		
	405	401	400
	es —	golden	es
with the particles	607,5	601,5	600

An extreme example:

Pythagorean 6q = ges 2 — =	588.3 cents	Syntonic fis + =	590.2 cents
Scheme 14: plus 24 k =	+ 34.4 —	Minus 9 k =	— 12.9 —
golden ges =	622.7 —	golden fis =	577.3 —
Scheme 21:		plus Molecule v =	+ 45.4 —
		ges =	622.7 —

b) **Scheme 15** shows the exact change (to three decimals) in the middle figure (15 tones) and gisis. The number of cents diminish or increase **evenly** with

$$\left\{\begin{matrix}5.741 \text{ cents in all horizontal Fifth-Axis} \\ 1.456 \text{ — — — oblique Third —}\end{matrix}\right\}$$

representing »the 2 building materials« in the syntonic mixing-system: Fifths and Thirds:

1) + 1.373 cents eis	2) — 2.912 gis	3) — 2.912 gis	4) 5 syntonic e = 378.2 feses +
— 1.456 plus e	+ 2.829 plus ♯	— 2.829 minus ♯	5 golden e = 356.8 feses
— 0.083 gisis	— 0.083 gisis	— 5.741 g	distance = 21.4 = 1 Comma

c) **Rationa lexplanation:** Super division of the octaves e ... e' and as ... as' (Supplement-pairs) gives »retrograde gisis« and »direct feses« respectively.

d) Similar methods (but of inferior structure) were up to 1558 used in the three famous systems of chance by: (see **Scheme 16**):

	Zero-Axis	The unit	
		increases	decreases
(X unknown author) before 1511......	a c es	+ 1/3 as	— 1/3 e
The Bohemian, Arnold Schlick 1511 ..	as c e	+ 1/4 a	— 1/4 es
The Italian, Gioseffo Zarlino 1558	ces c cis	+ 1/7 { as a	— 1/7 e — 1/7 es
here, Scheme 14 1930	feses c gisis	+ 1/15 as	— 1/15 e

Scheme 16 shows the 3 Fifths oscillating about the golden g, »the truth«:

	X	Zarlino	Golden system	Schlick
the result g:	694.8	695.8	696.2	696.6
by means of:	$^5/_{15}$	$^4/_{14}$	$^4/_{15}$	$^4/_{16}$
in decimals:	= 0,333	= 0,286	= 0,267	= 0,250
Distance from the ideal:	+ 0,066	+ 0,019	**Ideal**	— 0,017

Addendum.

Scheme 17 shows diagrams of 7 Minors and 6 Majors, some of them in pairs symmetrically, see pages 10, 15 and 20.

Glareanus's erroneous **nomenclature** of the medieval scale-type should once for all be obliterated from the litterature of music, and should be replaced by that recommended in Scheme 17, compare Helmholtz's heartfelt cry: »Übrigens werde ich Glareans Namen nicht brauchen .... es wäre überhaupt besser, wenn man sie vergessen möchte,« repeated by Ellis: »But I shall not use Glareanus's names .... It would be better to forget them altogether.« (Note 21).

Scheme 18 shows lines through corresponding tones in 5 temperaments.

Scheme 19: tertiary distances as arcs in a circle, see pages 7 and 16.

Scheme 20: construction of golden tones by means of parallel lines, see page 24.

Scheme 21: Golden tones in cents with 4 decimals, see page 24.

Scheme 22: examples of logarithms as training in the use of my Constant K = 0.600.5714, see page 7.

In **Scheme 23** is set forth a proposal for a system of notation on music-paper, staff, in all other tone-systems than the 12-toned temperament, with the suggestion, offered by K. Steensen (Note 22) for a rational arrangement of the ♯ and ♭.

Scheme 24 shows how the cents can easily be calculated by taking the difference between the logarithms for Partial-tones, the numbers of which are respectively the numerator and the denominator in a tone-fraction, according to the table constructed by the organist Kai Kroman, Copenhagen, for example:

oriental $d = \frac{10}{9}$ { Logarithm for Partial tone No. 10 = 3.986.3
— — — — — 9 = 3.803.9
difference ... 182.4 cents

The word »Partial-tone« is preferred to »over-tone« in order to avoid the **confusion**, which Ellis characterises as »great confusion, Tyndall's **erroneous** translation« (Note 21) between

the German ober	and the English upper: adjective .....
— über	— — over: preposition, adverb.

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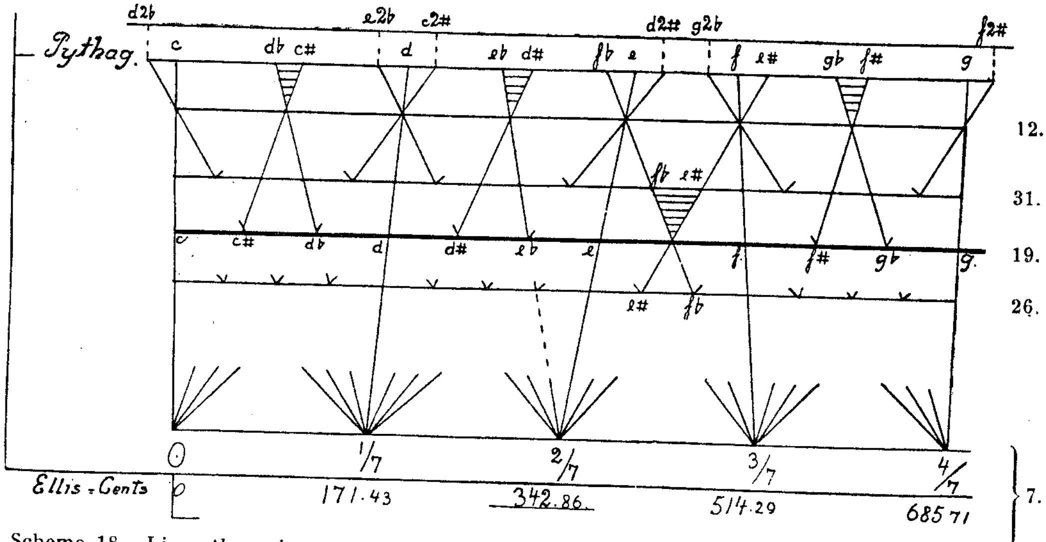
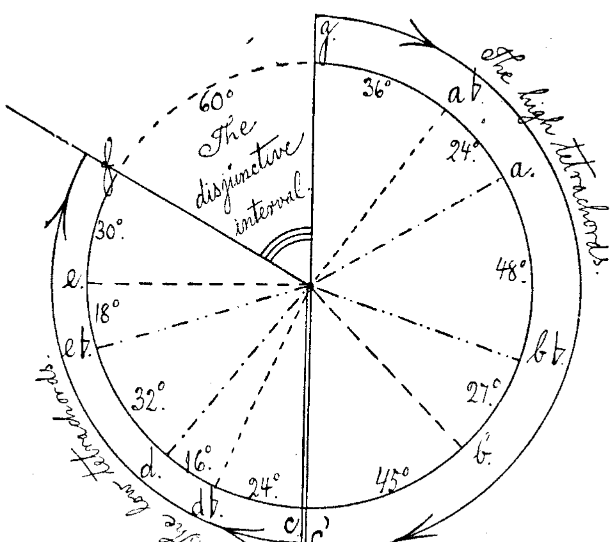
—	14	LUDWIG KAISER: »Über die Verhältniszahl des goldenen Schnitts«, Leipzig 1929, p. 122; THORVALD KORNERUP: »Die Hochteilung der Octave«, Copenhagen, Oct. 1930; the main contents is embodied in this treatise.
26	15	Phycicist HANS CHRISTIAN ØRSTED: »Samlede og efterladte Skrifter«, Copenhagen, B. 7. 1858, pp. 93—95.
28	16	Director GODTFRED SKJERNE: above mentioned »Plutarchos«, 1909, p. 77.
—	17	Professor JOSÉ WÜRSCHMIDT, Tucuman in Argentine: »Die rationellen Tonsysteme im Quinten-Terzen Gewebe«, in Scheel's Zeitschrift für Physik, Berlin 1928, p. 526, (in the German text tone »b« is used for the tone $^{15}/_{8}$); Spanish translation: »Los sistemas de sonidos racionales«, Buenos Aires, 1928, p. 3.
—	18	Professor JOSEPH YASSER: »A Theory of Evolving Tonality«, New York 1932, pp. 278—84.
—	19	Professor LOUIS KELTERBORN: »Die Quinten Spirale«, Darmstadt 1929.
29	20	Professor T. N. THIELE's »number-figures«, grafic statements by means of points of systems in »Beretning om Naturforskermødet« in Copenhagen 1872.
32	21	HERMANN V. HELMHOLTZ's »Tonempfindungen«, 1913, p. 441 and Ellis translation, »Sensations of Tones«, 1912, p. 269, rep. p. 25, Note.
33	22	Arrangement of ♯ and ♭ according to K. STEENSEN: »Den musikalske Skrivemaade«, in Skjerne's periodical »Musik«, Copenhagen 1922, p. 46.



Scheme 18. Lines through corresponding tones in the Pythag. system and 5 temperaments.



Scheme 19. Two Tetrachords tertiarily graduated as a semi-circle and a sector.