PART ONE

REASONS FOR THE ADVOCACY OF MULTIPLE DIVISION

The idea of using more than twelve tones in the octave, although many centuries old in the West, and a matter of everyday practice in certain sections of the Orient,[1] nevertheless appears radical to most Western musicians. In spite of the tremendous changes in musical taste and style which have taken place in recent years, very few musicians appear to see any general departure from the limit of twelve tones per octave as likely in the foreseeable future.[2] As believers in a cause recognized, therefore, as radical, the advocates of multiple division, however disparate their views, share a feeling of discontent with the prevailing system, 12-tone equal temperament. The objections these writers have raised to 12-tone temperament can be grouped into one or more of the following three categories:

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[1] Arabian music is generally considered to be based on a scale of 17-tones to the octave, while a 22-tone system is attributed to the Hindus. Neither system involves equal temperament.

[2] "The practice for the past five hundred years has favored almost exclusively systems with only twelve pitches in the octave. There seems no immediate prospect of that practice being discarded in favor of any system of multiple division." Barbour, op. cit., p. 132. Certainly this view is widely held.

1. Acoustic: wherein it is alleged that the intervals created in duodecimal equal division are deficient, usually because of their lack of agreement with small rational-number ratios. The thirds and sixths of our present tuning system are particularly subject to such allegations, but some writers also attack the sevenths, and a few even question the fifths.

2. Historical: wherein it is alleged that earlier musical systems based on other tunings possessed features which would be desirable today, but which have been lost through the adoption of 12-tone equal temperament. In general, the development of musical resources required earlier systems to expand (i.e., encompass a greater number of tones to the octave) or perish. They perished. But many writers would bring them back, expanded through multiple division.

3. Evolutional: wherein it is alleged that while there is not necessarily an inherent defect or inadequacy in 12-tone equal temperament, changes in musical style are working to render it inappropriate for further musical growth. Such a view leads an evolutionally oriented advocate of multiple division to examine historical precedents and acoustic bases for a new system. The emphasis remains on evolution, however, and the third group of writers takes a much less hostile view toward 12-tone temperament than do the other two groups.

A chapter will be devoted to each of the three categories listed above.

CHAPTER 1

THE ACOUSTIC BASIS FOR MULTIPLE DIVISION

Joseph Sauveur, the great deaf-mute mathematician and music theorist active around 1700, proposed that the best tempered system is that in which the octave is not divided in too many parts and in which the pure diatonic intervals are altered as little as possible.[1] In this short statement Sauveur posed several of the questions which have bothered musical theorists throughout the ages. How many tones are "too many"? How small an alteration may be considered "as little as possible"? It appears evident on face value that the more tones one uses in a tempered division of the octave, the more likely one is to find closer and closer equivalents to the pure diatonic intervals. Accepting Sauveur's basic premise for the moment, all temperaments can be seen as representing a balance between economy in the number of tones on the one hand and exactness of correspondence to "pure diatonic" intervals on the other.

Given Sauveur's "pure diatonic" scale as a basis, however, it is self-evident that some temperaments are far

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[1] "Le système tempéré le plus parfait est celuy qui ayant son octave divisée en peu de parties, a des différences les moindres et les moins inégales." Sauveur, Table Generale des Sistèmes Temperez de Musique, Histoire de l'Académie Royale des Sciences, 1711, avec les Mémoires de Mathématique et de Physique, page 315. The expression "pure diatonic" is taken from the article on Sauveur in Scherchen, The Nature of Music, page 42.

superior to others. 12-tone equal temperament is far superior to 11- or 13-tone temperament in the accurate representation of this scale. If one wishes to use as one's basis a system other than that which Sauveur called the "pure diatonic," it is possible that 11- or 13-tone temperament[2] might serve his purposes better than 12-. Once one abandons a single premise such as Sauveur's the possibilities become infinite. It is therefore necessary to begin by examining the building blocks of musical systems, the intervals, around which theories of consonance and dissonance, of scales, of tonalities, and, finally, of tuning systems, are devised.

THEORIES OF CONSONANCE AND DISSONANCE

Although the terms "consonance" and "dissonance" tend to suggest specific understood phenomena to musicians, and although they are essential elements in almost all theories of music, they have been interpreted in varying ways by different authorities. There are those who define consonance and dissonance in primarily emotional terms, the one as a source of pleasurable sensation, the other as a source of a disquieting sensation. Others prefer to define consonance and dissonance in the context of the dynamics of

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[2] Ernst Krenek has used 13-tone equal temperament for a special musical effect in an electronic musical interlude in his _Spiritus Intelligentiae Sanctus_. Musical Quarterly, April, 1960, page 219.

a musical phrase, consonance representing a state of "rest" at the end of a phrase, dissonance a state of urgency and a search for "rest." Still others have defined consonance by the relative position of two tones within a scale without regard to specific harmonic considerations.[3] However, the most frequently used, and probably the most universally accepted, definitions of consonance and dissonance rest on one of two assumptions. The first of these is that consonance is produced by agreement between at least some of the component parts of two or more simultaneous musical sounds; this agreement is the result of their fundamentals creating a small-number ratio, such as 2:1, 3:2, etc.[4] The second is that, while indeed certain combinations of sound are consonant and others dissonant regardless of their musical context, it is the conditioning of the ear rather than the inherent qualities of the mathematical ratio that makes them

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[3] Joseph Yasser maintains that the interval between the first and third members of any functionally used scale is a consonance, regardless of the number of tones in the scale. A Theory of Evolving Tonality, page 197.

[4] It need not be dwelt upon that various acousticians explain the agreeable or allegedly agreeable effects of small-number ratios in various ways. Whether it is because of the absence of beats (Helmholtz), the melting together (Verschmelzung) of the individual sounds (Stumpff), or the formation of agreeable and comprehensible rhythmic patterns by the combined sound waves (Opelt), a simple ratio between the concomitant parts is found by all these writers to be the phenomenon which causes a combination of sounds to be consonant. This alone is what matters in considering the implications of the doctrine of natural consonance on systems of tuning.

so.[5] The writers who adhere to the first view might be called members of the "Natural Consonance" school, while those who adopt the second might be considered as believing in "Conditioned Consonance."

Concerning the supporters of conditioned consonance who advocate multiple division, they can hardly base any preference for change on acoustic grounds, since the consonances they recognize are the "consonances" in current use, the 700-cent fifth and the 300- and 400-cent thirds. Believers in conditioned consonance can, of course, support multiple division on the grounds that the ear will eventually become conditioned to the "consonances" of a new system. However, many opponents of proposed systems of multiple division have used, in effect, the argument that the ear has been conditioned to prefer "consonances" as they are.[6] It has been realistically observed that if indeed preference is a matter of conditioning, progress from one tuning system to another must overcome almost insuperable obstacles. The acoustical basis for multiple division

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[5]Harry Partch attributes to Aristoxenos of Tarent the first direct statement of the doctrine that the ear, and not the small number ratios, is the determinant of consonance. Partch argues, however, that "the results by ear and by small-number ratios are of course identical." Genesis of a Music, page 239. Helmholtz is probably the most renowned of the many theorists and acousticians who have advocated natural consonance.

[6]It has been the thirds in particular where believers in conditioned consonance have argued against systems which closely approximate the just ratio 5:4. See below, page 26.

is therefore cited almost exclusively by advocates of natural consonance.

Further complicating the picture is the fact that many writers accept limited portions of a theory of natural consonance while rejecting others. As Partch rightly points out, even those who would discount the idea of the natural validity of the small-number ratios use the most basic small number ratio, 2:1, as the basis for their musical systems.[7] There have been many theorists who have shared the basic view of Pythagoras that octaves and fifths should be represented by small-number ratios but that thirds should not be. The particular qualities of 12-tone temperament have tended to reinforce such a view. There have been many theorists who have maintained that consonance is only possible where the numbers in the ratio were 6 or smaller.[8] Others have accepted as consonant such ratios as 10:9 while refusing to consider the prime number 7 as a factor in musical consonance at all.

It is evident that for many writers who accept small-number ratios in principle, the question "when does a small number become a large number?" must be answered. The

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[7] _Op_. _cit_., page 239.

[8] Zarlino was, of course, a most famous exponent of the theory of the senario as the basis for consonance. From the Renaissance to the 19th century, the view of the senario as the basis for consonance seems to have prevailed overwhelmingly over both those who would reduce to 4 and those who would expand to 7 or more the number of natural partials used in generating a musical system.

subsequent parts of this chapter, working upwards from the octave, examine the answers which have been given to these questions.

One other aspect of the general relationship of consonance and dissonance needs to be considered. There has been increasingly strong controversy in recent decades over whether consonance and dissonance should be considered as opposed polar forces at all. The traditional view, enunciated emphatically by Zarlino and maintained for several centuries, accepts the polarity of consonance and dissonance as fundamental to music. Many modern theorists have opposed this view. Arnold Schönberg asserts that so-called dissonant intervals are nothing more than the less simple consonances.[9] The practice of 20th century composers seems to indicate support for Schönberg's view.

Among disciples of multiple division, opinions on the polarity of consonance and dissonance are mixed. One of the major proponents of 19-tone temperament, writing under the pseudonym Ariel, espouses multiple division on the grounds that it is necessary for the restoration of polarity between consonance and dissonance which has become corroded.[10] On the other hand, Ivan Wyschnegradsky, a

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[9] "Dissonanzen sind die fernerliegenden Konsonanzen." Cited in Yasser, A Letter from Arnold Schönberg, AMS Journal, Spring, 1953, p. 53 ff.

[10] Ariel, Das Relativitätsprinzip der Musikalischen Harmonie, p. 148.

leading advocate of 24-tone temperament, considers the abolition of consonance and dissonance as polar opposites to be a basic starting point for his system.[11] It is understandable that a system of multiple division seeking to exploit a "twilight area" between consonance and dissonance or to abolish consonance and dissonance altogether, will differ from a system attempting to reinforce the polarity of the two elements.

Intervals:

THE OCTAVE

"It is universally agreed that the octaves should be exact," wrote the Englishman R.H.M. Bosanquet in 1876 in his Elementary Treatise on Musical Intervals and Temperament.[12] There was much justification for his remark when it was made, and it is quite imaginable that the same statement might be made today, although the word "universally" would have to be changed. The octave is alone among the small-number ratios in retaining its special characteristics in the musical hierarchy in the aesthetics of dodecaphony, and it remains the basis for almost every proposed musical system. The uniqueness of the octave has been described many times, but why it should have such a unique effect

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[11] Wyschnegradsky, Quartertonal Music, its Possibilities and Organic Sources, Pro Musica Quarterly, Oct. 1927, page 30.

[12] p. 2.

appears as unanswerable as why green should look green.[13]

In this century, a few musicians have begun to build systems which do not recognize this hitherto conceded hegemony of the octave. A typical and noteworthy case is Stockhausen's Studie II, where the interval 5:1 is chosen in preference to 2:1 as the basis for an equal temperament.[14] The interval 5:1 is divided into 25 equal parts. Other Stockhausen works have involved other systems in which the octave is not to be found. In some works of the Cologne school the continuum of pitch plays a role secondary to the continuum of saturation (at one end of which is the sine-wave, at the other end, noise) in which no phenomenon similar to the octave is discernable. Such experiments having been started, it is conceivable that the octave may begin to play a diminishing role in the formulation of musical systems.

Ivan Wyschnegradsky, who used musical systems based on the octave, has proposed scales in which pitches do not

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[13] In comparing the audible sound continuum with the visible spectrum one is struck by the fact that while the latter proceeds directly from red to indigo without at any point in the interim suggesting red again, the pitch qualities appear to repeat themselves every octave. C returns, likewise E, G#, etc. Red never returns. There are in all about 10 recurrences of each pitch class within the limits of the audible range of frequencies.

It is possible that the ratio 2:1 would have the same effect visually that it has aurally. We do not know for sure because the human eye perceives only a color range extending from about 4000 to about 7700 angstrom units. But the tendency of red and blue to unite in violet suggests a possible visual quality to the 2:1 ratio not unlike the effect of the octave.

[14] Die Reihe, Bd. 1, footnote to page 50.

repeat themselves at the octave but rather at the second or third octave.[15] Nicolas Slonimsky includes in his Thesaurus of Scales many which involve repetition at the second or third octave only, or even at a still wider interval. Joseph Yasser at one time suggested an incursion on the hegemony of the octave by treating the perfect fourth or the perfect fifth as a complete identity capable of enclosing a musical system.[16] Such a fourth and fifth he proposed to call a "quartave" and a "quintave" respectively. The tetrachord relationships of the Greek Lesser Perfect System might be considered based on such a quartave principle, and it is possible to view parallel organa and some 20th century fourth doublings as a partial manifestation of a similar principle.

In addition to proposals to eliminate the octave or to replace it as the primary "identity" in music, there have been a few suggestions of recent origin that the octave should not be tuned precisely.

A number of studies have shown that at the highest and lowest ranges of hearing the interval 2:1 must be stretched a substantial amount to produce the psychological effect associated with the octave. The suggestion has recently been made that 2:1 be slightly enlarged throughout

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[15] Manuel d'Harmonie À Quarts de Ton, page 15.

[16] The Highways and Byways of Tonal Evolution, AMS Bulletin, Sept. 1948.

the pitch continuum to produce its most agreeable effect.[17]

It has also been proposed that the octave be made smaller. Adriaan Fokker, one of the great contemporary figures in the field of multiple division, has asserted that the octave should not be regarded as the inviolate basis for a musical system but rather as one of several necessary consonances each of which is equally important. The unit in his temperament represents a mean between rational fractions of the octave, the perfect 5th, perfect 4th, major 3rd, minor 6th, natural 7th, and "super" 2nd. Every one of these intervals is altered so that a minimal total alteration shall be necessary.[18] Example 2 shows the factors in Fokker's determination of the proper size for a unit in 31-tone temperament.

Example 2: Factors in Determining the Unit for 31-Tone System

39.0 cents is 1/18 of 3:2
38.2 cents is 1/13 of 4:3
38.6 cents is 1/10 of 5:4
38.8 cents is 1/21 of 8:5
38.8 cents is 1/25 of 7:4
38.5 cents is 1/6 of 8:7
38.7 cents is 1/31 of 2:1
38.65 cents is the weighted mean.

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[17] Kolinsky has proposed that every octave be enlarged by nearly two cents. AMS Journal, 1951. According to Wilhelm Dupont, a survey by Stumpff and Meyer reported in the *Zeitschrift fur Psychologie and Physiologie der Sinnesorgane*, Bd. XVIII, heft 5 & 6, discovered that 79% of those sampled preferred octaves larger than the pure 2:1. *Geschichte der Musikalischen Temperatur*.

[18] Fokker, *La Gamme, La Musique, et le Tempérament égal*, 1951, page 147.

In calculating his mean value for a unit of the system, Fokker gives the octave approximately double the weight of each of the other units, and he gives the smaller of the remaining intervals greater weight than the larger ones on the grounds that a slighter change produces more pronounced effects where the interval is smaller. Fokker's octave is 1198.2 cents, just about as shortened as Kolinsky's is lengthened. "Perhaps it might be better to choose that value for the basic interval," suggests Fokker,[19] in a statement which certainly falls short of militancy.

In his charts he also represents the octave as one dimension of a four-dimensional scheme, with unity representing the sum of the squares of the member dimensions.[20]

Despite occasional suggestions to disregard the octave, to enlarge the octave, or to shorten the octave, its role in music remains so assured that it appears to be in little danger of going into eclipse. Among the many proposed changes in musical nomenclature brought into existence by proponents of multiple division and by others, there does not appear to be a single recommendation to reletter the notes so as to deny the existence of an octave identity.

Even the vigorous serial movement, which treats all

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[19] "Peut-être vaudrait-il mieux choisir cette valeur-là pour l'intervalle elementaire du tempérament." *Ibid.*

[20] Fokker, *Les Mathématiques et la Musique*, p. 28.

intervals within the octave as equal, recognizes in the octave its traditional value as replica or identity. The recent debate within the ranks of dodecaphonists over whether to permit octave doubling within the "row" suggests that to some, at least, the existence of the octave as an interval outside of the scope of serial treatment, is an enigma, its special treatment creating possibly some discomfort. But the total freedom of melodic octave-displacement within dodecaphony demonstrates further that the octave is not subject to the laws of the other intervals within the system.

In light of the above it appears reasonable to anticipate the continuance of the octave and to stipulate its use in the systems of multiple division which will be explored below. It is also a practical necessity to assume that the addition or subtraction of an octave does not change the essential character of an interval. Otherwise it would be necessary to consider the octave-and-a-fifth (the twelfth) separately from the fifth, etc.

THE 3rd PARTIAL: THE PERFECT FIFTH AND FOURTH

"The strongest consonance in music next to the 2:1," is how Harry Partch characterizes the 3:2 or perfect fifth.[21] This statement by one of the most eminent opponents of present-day tuning procedures accords with the

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[21] *Op. cit.*, pp. 304-5.

principles set forth by one of the leading protagonists of a tuning approaching 12-tone equal temperament when it was new and controversial. Said Werckmeister, "The nearer a ratio is to unity, the more perfect is its consonance."[22] The perfect fifth is, after the octave, the nearest possible interval to unity. All of the known tuning systems of the Western world have taken the perfect fifth into consideration and have rendered it with a degree of accuracy which, if questioned by some authorities, has been higher than the accuracy consistently attained by any other interval except for the octave.[23]

As to whether the interval 3:2 is common to all of the world's musical systems, as has occasionally been claimed, Fritz Kuttner asserts that the "fifth" in Chinese music is 20 to 30 cents flat.[24] It is apparently nearly as flat in Siamese music if, as is generally agreed, Siamese

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[22]"Je näher eine Zahl der Unität, je vollkommener dieselbe ist," cited by Dupont, op. cit., page 76. Many historians including Dupont ascribe to Werckmeister an important role in the establishment of 12-tone equal temperament. Barbour doubts, however, that Werckmeister was, in fact, an advocate of equal temperament, claiming instead that he advocated only a "well-tempered" system in which modulation to any key was possible while the keys retained their individual characteristics.

[23]In the meantone tuning system the third is tuned perfectly at the expense of the fifths. This tuning system was, however, of short duration and its adoption was never universal. Its position of favor was probably less influenced by a preference for the third over the fifth as such than by the simple fact that only a quarter as much discrepancy in the fifths was necessary for a perfect third as the other way around.

[24]Communicated by discussion, December 1960.

music approximates 7-tone temperament with its 685.7-cent fifth. Nevertheless, the interval 3:2 has been an essential part of Chinese musical theory for many centuries, and the interval is a strongly felt part of all highly developed systems of the West, Near East, and India. Alain Danielou, a comparative musicologist, has called the 3rd partial the "cyclic" element in music, by which he presumably means that the domain of musical systems is generally defined by cycles of fifths.[25]

To Bosanquet, the fifth is the basic interval of any musical system, and "regular systems are such that all their notes can be arranged in a continuous series of equal fifths."[26]

Recognition of the primacy of the fifths among intervals within the octave extends to those who would temper it considerably (and even, paradoxically, to one who would abandon it). Robert Smith, an 18th century English theorist, in giving tuning instructions, indicates that choosing a temperament for the fifth is a most essential step.[27] Joseph Yasser, a contemporary advocate of 19-tone temperament, considers the cycle of perfect fifths to be a "common

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[25] Daniélou, Introduction to the Study of Musical Scales, 1943, pages 230-231.

[26] Op. cit., p. 60.

[27] Smith, Robert, Harmonics or the Philosophy of Musical Sounds, 1759. He would alter the fifth by more than 6 cents.

denominator" for all musical systems. He first produces his 19-tone scale as a series of fifths with a comma of 138 cents.[28] This he does, despite a remarkable decision not to accept 3:2 as the basis for a consonance in the 19-tone system at all.[29]

With the exception of this suggestion from Yasser, surprising in light of his use of the perfect fifth in deriving his system, and with similar exception of a few unique temperaments, generally proposed only for single pieces,[30] the fifth remains an integral part of intonational systems. From the rise of medieval modes through the end of the 19th century this interval has unquestionably played what is correctly called the "dominant" role, harmonically and structurally, in our music. The dominating interval in the "rule of the octave" which so concerned Rameau and his contemporaries, the fifth defines both the authentic and plagal cadences, and its primacy among intervals other than the octave appears to be fundamental to functional harmony.

Because of the mathematical-acoustic basis for the use of the perfect fifth as dominating interval, apologists

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[28] Yasser, A Theory of Evolving Tonality, 1933, page 116.

[29] Ibid., pages 176 and 285. For further discussion see Chapter II, below.

[30] Such as those works of Krenek and Stockhausen which are cited above, pages four and ten respectively.

for the various schools of 20th century composition which have chosen not to use the fifth above other intervals as a source for harmony or tonal progression have been hard put to justify this aspect of their case. Allen Forte, in the introduction to his book, Contemporary Tone Structures, states, "Much of contemporary music is rendered invalid from that point of view (that the pre-eminence of the fifth is naturally ordained)," and he asserts that the overtone series as a natural basis for the dominant-tonic relationship is open to question on several points.[31]

That nature may, however, play a role in the long-manifested human preference for the fifth is evidenced by the simplicity of its ratio and attested by the continuing body of writers who select that interval as an important basis for their musical systems. It would appear that any system of multiple division based on the principle of the consonance of the small-number ratios would have either to include the perfect fifth or to offer a satisfactory explanation for its absence.

The Fifth: Pure or Altered? It remains to be determined whether the perfect fifth shall be pure or altered, and if altered, to what extent. Probably the tuning system with the longest history in the West is the so-called Pythagorean system based on absolutely pure fifths. This

[31]Forte, Contemporary Tone Structures, pp. 10-11.

system, in general use during the Middle Ages, probably employed extensively in antiquity, and used by the majority of string players today,[32] can never be closed perfectly since no number of exact fifths can ever equal an exact number of octaves.[33] Here lies one of the primary defects of Pythagorean tuning, the imperfect relationship between fifth and octave. In the other direction lies its other imperfection, the imperfect relationship between the fifth and other possibly desirable small-number ratios. Just as no power of 3 can ever equal a power of 2, neither of them can ever equal a power of 5, or of 7 etc.

When operative in a 12-tone system, Pythagorean intonation leaves a discrepancy of 23.5 cents, representing the ratio 531441:524288, which is known as the Pythagorean comma. As mentioned above, when applied to a 19-tone system, Pythagorean tuning leaves a discrepancy of about 138 cents, a figure equal to more than 2 units of the tempered 19-tone system. Much less is the discrepancy in a 17-tone system and many theorists believe the Arabic 17-tone system to be Pythagorean.[34] Yasser has offered an

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[32] For a strong statement of its importance by a disciple of its role see Barbour, The Persistence of the Pythagorean System.

[33] This is obvious in the fact that all powers of two must be even numbers, while all powers of three must be odd numbers.

[34] According to Partch, early Arabian musicians such as Zalzal used non-Pythagorean systems but 14th century theorists simplified to Pythagorean tuning which has remained

interesting postulate which derives support from the apparent preference of much of antiquity for Pythagorean tuning. He suggests that it is in the nature of all pentatonic music to eschew the use of the just third, making Pythagorean tuning the "just intonation" for the 5-tone scale.[35]

Among the theorists who have advocated temperaments for the fifth, some have done so reluctantly and sparingly, but others have claimed to prefer the sound of the tempered fifth to that of the pure fifth. Among the temperaments adopted or proposed, our present system involves relatively slight altering of the fifth. Nevertheless, there have been those who have based their search for a better equal temperament on the postulate that the fifth must be even better than the one found in 12-tone equal temperament.[36] A far greater number of advocates of multiple division have been

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since. Helmholtz however prefers to regard the Arabic system as comprised of 9 fifths with 8 tones added a third above all but one of the initial 9. Owing to the close agreement between the major third and the interval comprised of 8 perfect fifths, an agreement which Helmholtz records in the theorem bearing his name, the difference between Helmholtz' explanation and the more usual one involves only a minute gradation in pitch.

[35] *Op. cit.*, page 131.

[36] Jankó, a pianist and theorist who is best known for his attempt to popularize a reform in keyboard design, considers the question of multiple division, and makes the initial stipulation that the fifths must be better than those of 12-tone temperament, leading him to a choice between 29-, 41- and 53-tone temperaments, of which he chooses the second. *Über mehr als zwölfstufige Temperatur*, Beiträge für Akustik und Musikwissenschaft, heft 3, p. 6.

willing to see the fifth altered more than in 12-tone temperament in order to improve other intervals in the temperament. By a shortening of the fifth by 5½ cents, a temperament containing the just major third is achieved, while a fifth which is 7 cents too small produces a system with a just minor third. The specific systems which are created by the various alterations in the size of the fifth will be discussed in later chapters. Relevant to the present discussion is the evaluation of the alterations in the fifth itself, as offered by the various writers.

On the one hand there is Partch who claims that even 2 cents is an excessive alteration.[37] On the other hand there are numerous advocates of multiple division who judge deficiencies of about 7 cents to be within the range of tolerance.[38] None of the systems of multiple division which have obtained broad support possesses fifths with a greater discrepancy than this, suggesting about 7½ cents as an implied limit for the fifth. This is a far smaller figure than that of the discrepancy of other supposed natural consonances in 12-tone temperament.

There are theorists who maintain that the perfect

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[37] "A heard 2-cents discrepancy is a sensible difference in simultaneous soundings." _Op. cit._, p. 310. For this reason, Partch opposes all equal temperaments.

[38] Ariel asserts that the 7-cent deficiency of the fifth in 19-tone temperament is well within the limit beyond which consonance is destroyed. "Sie bedeutet aber noch lange nicht die Grenze für die Trübung des Wohlklanges." _Das Relativitätsprinzip in der Musik_, p. 144.

fifth as the principal interval within the octave requires a purer tuning than other consonances. Against this is raised the counterclaim that the more complex and subtle consonances are more easily destroyed than the fifth by slight alterations in tuning. Fifths in practice as well as theory have been the least distorted intervals after the octave.

It should be noted, however, that there have been occasional theorists who have preferred altered to unaltered fifths. Robert Smith, in the 18th century, defending mean-tone principles against the increasing tendency toward 12-tone equal temperament, enunciated the principle that if indeed the consonances of a musical system must be tempered, it is best that they all be tempered moderately rather than that one or two should have to be tempered a great amount in order that other consonances might be almost pure. Evaluating all intervals from this standpoint, which he called equal harmony, Smith attacks the fifths of 12-tone temperament as being "finer than they ought to be."[39] The ideal fifth, according to Smith, would be 5/18 of a comma smaller than 3:2. This is almost exactly six cents. M. W. Drobisch, a 19th century theorist, uses methods similar to Smith's in finding a value for the fifth that will work best

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[39] Smith, Harmonics, p. 166. Sauveur had been concerned with the equalization of the discrepancies of the intervals as well, as can be seen from the quotation in footnote 1 on page 3, but he was content with an alteration of only about 4 cents in the fifth.

with the other consonances of the diatonic system and also selects a flat fifth, 697.2 cents.

In the 20th century, the Danish mathematician and acoustician Thorvald Kornerup has proposed a new system of acoustic valuation aimed at interrelating all of the consonant intervals in a single mathematical progression. As part of his over-all plan, the "ideal" fifth is to be 696.21 cents, nearly 6 cents smaller than 3:2. To support his judgment favoring the smaller fifths, Kornerup relates the 3rd partial to each of the higher partials through number 32. Example 3, below, shows how many fifths must be superimposed to produce each of the partials from 9 to 16, and the size of each of the fifths in order that the partial produced should be pure.[40]

Example 3: Kornerup's Relationships of Fifths to Partials

NUMBER OF FIFTHS	SIZE OF EACH FIFTH	PARTIAL PRODUCED THEREBY
2	701.95 cents	9
4	696.58 cents	10 (or 5)
1/13*	696.05 cents	11
1	701.95 cents	12 (or 6 or 3)
15	696.03	13
10	696.88	14 (or 7)
5	697.65	15
12	700.00	16 (or 8 or 4 or 2 or 1)

* = thirteen fourths rather than fifths: the exact inversion.

[40]Kornerup, Acoustic Valuation of Intervals, 1938, page 13. For a more detailed examination of Kornerup's theories see Chapter 9.

As can be seen, the fifths are all either 701.95, the size of the perfect fifth, or smaller, with most of them in the region 696-697, where Kornerup's "ideal" fifth lies. The table appears to give support to the view that in an equal tempered musical system attempting to give close approximation to the upper partials, the fifth will have to be reduced in size, perhaps substantially. However, Kornerup's methods in arriving at his figures are not above criticism. He appears to have excluded fifths larger than 701.95 quite arbitrarily, even in cases where a shorter cycle of fifths closer to 701.95 would have produced the same partial. For example, in the case of the 13th partial, Kornerup requires no less than 15 fifths, each of which is nearly 6 cents too small. However, if he had substituted a fifth only three cents too large, 705.0 cents, he would have been able to produce the 13th partial with a series of but 8 fifths (705 x 8 = 5640, minus 4800 = 840 cents = 13:8). Kornerup's table remains informative in showing the various amounts of alteration necessary to relate the shortened fifth to other members of the harmonic series, but its validity as a justification for his theory of the foreshortened fifth is severely weakened by the <u>a priori</u> exclusion of the lengthened fifth.

How to evaluate theories giving actual acoustical preference to the foreshortened fifth? Since they all involve relationships between the ratio 3:2 and other numbers, one is tempted to disregard theories of a small fifth altogether as externally ordained and to question strongly whether any of the authors, given just a fifth, really prefer the foreshortened version. Nevertheless, in view of the unquestionable preference of many contemporary writers for enlarged major thirds over pure 5:4's, the genuine preference of some musicians for an altered fifth is quite possible. In Smith's case, recalling the prevalence in his day, especially in England, of mean-tone tuning involving a distinctly flattened fifth, it is quite possible that conditioned ears or innate conservatism quite convinced him of the acoustical superiority of the small fifth. Certainly if we are to accord a measure of acoustical validity to the treatment of the enlarged major third as a consonance, it would appear only reasonable to accord similar regard to the preference for the foreshortened fifth, especially as the extent of tempering in the latter case is much the smaller, not exceeding the 7.21 cents called for in 19-tone temperament. For the purposes of this paper, all intervals within this range of 3:2 will be treated as representatives of the perfect fifth, and the perfect fifth will be regarded as an essential part of any system of multiple division, with 701.95 cents regarded as the ideal size.

THE 5th PARTIAL: THIRDS AND SIXTHS

With the addition of the 5th partial, four intervals are added to the field of basic consonances: 5:4, the major third; its inversion, 8:5, the minor sixth; 6:5, the minor third, representing the difference between the perfect fifth and the major third; and 5:3, the major sixth, which is the inversion of the minor third. These four intervals, represented in cents, are approximately 386, 814, 316, and 884, respectively. Because every one of these intervals deviates from its equivalent in 12-tone temperament by well over half a syntonic comma, it is on the subject of the fifth partial that the advocates of 12-tone temperament have been historically most on the defensive. Rather than offer the argument that a discrepancy of 13 to 15 cents is negligible, defenders of 12-tone temperament have often contended that a large major third is preferable to a just third. Many string players and singers are taught to exaggerate the size of even the major thirds of 12-tone temperament. As the argument over the proper size of the thirds, and whether they should in fact be considered as representatives of a consonant 5th partial, is basic to nearly all questions related to multiple division, the points made by both sides will be examined in some detail.

Daniélou, who considers the third partial to be "cyclic," calls the fifth partial "modal," describing it further as the "humanizer" of music.[41] But just as aestheticians disagree on how extensive a role the 5th partial should play in today's music, historians and musicologists differ on the role of the just third in other ages and civilizations. Ratios containing the number 5 exist in the theoretical writings of Ptolemy near the end of the Greek era. It is generally agreed that throughout the early middle ages the fifth partial was not used. However, there is disagreement about the time and extent of its re-introduction into musical theory and practice, and this disagreement closely parallels the disagreement concerning its aesthetic validity.

J. Murray Barbour, a leading expert on tuning systems and a strong partisan for 12-tone temperament, has claimed on several occasions that the first re-introduction of the 5th partial to musical tuning systems occurs in the writings of Bartolomeo Ramos de Pareja in 1482. Thorwald Kornerup and other theorists who have valued the 5th partial more highly than Barbour ascribe to Walter Odington, an English theorist active around 1300, the re-introduction of this partial as a means of explaining the relative consonance of the third. Barbour acknowledges Odington's "discovery" of the interval 5:4 but apparently does not believe that it was used by him or by his contemporaries as part of a

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[41] _Op. cit._, page 231.

complete musical system.[42]

Building their case around Odington's alleged advocacy of the 5:4 ratio in explaining the third, and around the English practice known as gymel, wherein parallel thirds and sixths were sung as consonances not unlike the later French faux bourdon, the supporters of the 5:4 claim to see a connection between the beginnings of the theory of the just third and the beginnings of the treatment of the third as a consonance. The third in Pythagorean music, not being based on a small number ratio, was a dissonance and was treated as such.[43] If, indeed, the general acceptance of the third as a consonance corresponds in time to the adoption of 5:4 as the correct intonation of the third, important historical support is given to the case for the just ratio. The opposing claim is that the interval 5:4 was introduced after the consonance of the third had begun to be established and that just thirds never did achieve the widespread use that has sometimes been ascribed

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[42] In Tuning and Temperament, Barbour cites Odington's reference to 5:4 and 6:5 as the consonant thirds on p. 3, but chooses Ramos as the founder of just intonation because he was "the first known European writer to break away from the Pythagorean tuning for the tuning of the chromatic monochord," p. 89.

[43] Yasser considers the major third to be a dissonance in all pentatonic music, which, he contends, is based on Pythagorean tuning. A Theory of Evolving Tonality, p. 111. Kurt Blaukopf, a respected musico-sociologist who has been greatly influenced by Yasser's writings, makes the same assertion. Musiksoziologie, p. 62.

to them.[44]

The preference for the just third among theorists and acousticians, especially in the 19th century and shortly before, is quite extensive. Helmholtz sounds the keynote when he says, "The principal fault of our present tempered intonation lies in the thirds, and . . . is the old Pythagorean error of forming the thirds by means of an ascending series of four fifths. The natural relation of the major third to the tonic, both melodically and harmonically, depends on the ratio 5/4 (5:4) of the pitch numbers. Any other third is only a more or less unsatisfactory substitute for the natural major third."[45]

Bosanquet calls the Pythagorean third "disagreeable,"[46] after asserting that "chords formed by the notes ordinarily in use are much inferior in excellence to chords which are in perfect tune."[47]

In this century many of the leading theorists and acousticians have continued to assert the superiority of the natural third. Carl Eitz, at the beginning of the century, asserted that "A good a capella chorus sings, as far as a composition permits, in just intonation, since every

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[44]Op. cit., p. 89.

[45]Helmholtz, Hermann L. F., On the Sensations of Tone as a Physiological Basis for the Theory of Music, translated from the German by Alexander J. Ellis, p. 315.

[46]Bosanquet, op. cit., p. 8.

[47]Ibid., p. 6.

normally gifted singer has an excellent sensitivity for pure octaves, fifths, fourths, thirds, and sixths. . . ."[48] A similar view has more recently been expressed on a number of occasions by Lindsey Norden and by many others.

Even among some writers well disposed toward 12-tone temperament, the 5th partial is recognized as the source of the consonance of the thirds. Rameau in the 18th century is a good example, as he unqualifiedly accepts the senario as the basis for consonance but is not ill-disposed toward 12-tone temperament as a compromise. Hindemith's views on the 5th partial and on 12-tone temperament are similar.

Others, however, have voiced an emphatic preference for the thirds of 12-tone temperament over the thirds of the unaltered 5th partial. In the notes which accompany the first release in Musurgia Records' series on the History of Musical Theory, Fritz A. Kuttner and J. Murray Barbour state,[49] "As will be seen and demonstrated in the recording, there is . . . reason to doubt the musical merit of a superparticular ratio as low as 4:5." In the introduction

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48."Ein guter A capella-Chor singt, soweit es die Komposition zulässt, in natürlich-reiner Stimmung, denn jeder normal veranlagte Sänger hat ein feines Empfinden für reine Oktaven, Quinten, Quarten, Terzen und Sexten. . . ." Eitz, Carl, Von den natürlich reine Stimmungsverhältnissen, Melos I, 1920, p. 293.

49Kuttner, Fritz, and J. Murray Barbour, The Theory of Classical Greek Music, Introductory Notes, New York, 1958, Musurgia Records, Column 10.

to their next release, The Theory and Practice of Just Intonation, the same two authors state that a major defect in just intonation is "boredom with the insipid major and minor thirds that are too perfect for our ears to have much character."[50]

A great number of writers have taken a middle position with respect to the thirds by endorsing the just third for harmony while preferring a larger third for melody. Alfred Jonquière, a notable acoustician whose Grundriss der Musikalischen Akustik of 1898 proved serviceable as a standard work for many years, takes this position on the basis of experiments which showed that musicians prefer the large major third for melody but the small one for harmony.[51] Engbert Brandsma, reporting to the congress of the Internationale Musikgesellschaft in 1909, follows the same path as Jonquière, calling the melodic, Pythagorean intervals "structural" while considering the harmonic, pure intervals a source of "color." Brandsma uses several musical examples to illustrate his point, the first being Agatha's theme from Der Freischutz (Ex. 4), in which the interval 5:4 above the tonic for the E would, in Brandsma's

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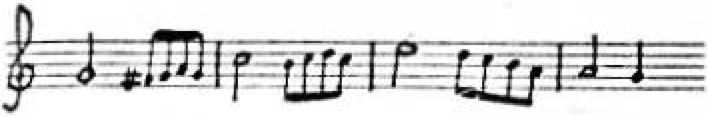
[50] Kuttner and Barbour, The Theory and Practice of Just Intonation, Introductory Notes, column 31.

[51] Jonquière, Grundriss der Musikalischen Akustik, p. 139.

opinion, be "expressionless."[52]

Example 4:



In spite of its apparent harmonic derivation, Brandsma insists similarly that the F# in the opening theme of the finale of Beethoven's violin concerto must be Pythagorean in order to achieve sufficient brilliance. The folksong "Sleep, Baby Sleep," however, is best sung with a just third in order to achieve repose.

Writing in 1955, Thomas P. Frost reacts in much the same way as Jonquière and Brandsma to tapes of Eivind Groven's organ[53] in enlarged just intonation.[54] He concedes that his reaction may be due in part to conditioning: "A musician accustomed to equal temperament will find many of the scale tones, particularly the 3rd and 6th, so far removed from their accustomed pitch as to seem 'sour.' When a scale or solo melody is played, it may appear disconcert-

[52] Brandsma, Uber die Tonverhältnisse in der alten und neuen Musik, Bericht von III Kongress der Internationalen Musikgesellschaft, p. 353 ff.

[53] Eivind Groven is a Norwegian theorist and composer whose 36-tone organ is discussed below in this chapter and in Chapter 7.

[54] Frost, Thomas P., A Matter of Records, Organ Institute Quarterly, Vol 5:3, Summer 1955.

ingly out of tune." Frost adds, however, that "When these same notes are combined into chords, the sounds are magically consonant. The complete and unaccustomed absence of beats between the notes (makes them) all but blend into one tone. The pure richness of the untempered chords is a novel experience which we wish it were possible to share with every reader."

String players are generally instructed to play sharped notes somewhat higher than the enharmonically equivalent flatted notes. This practice tends to produce Pythagorean thirds. Helmholtz, however, informs us that Joachim played just thirds on all occasions,[55] and Dupont does not question this assertion in reporting it in his history of temperament.[56] Arguments persist to this day as to whether singers use just or large thirds when working away from a piano. It appears that more and more authorities are tending to doubt that just thirds enjoy any significant use.[57]

The minor third, 6:5, has stirred up some controversy

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[55] Helmholtz, op. cit., p. 323.

[56] Dupont, op. cit., p. 130.

[57] It is Kuttner's preliminary finding in general research supported by stroboscopic measurements that the intonation of the different musical media, piano, strings, voices, etc. is far less varied than is generally thought. According to Kuttner, a well-tuned piano tends to have slight deviations from equal temperament (perhaps 2 or 3 cents on the average), while voices and strings approach equal temperament nearly as closely. --conversation, December 1960.

of its own. Ranked as a subordinate interval to the major third by Descartes, because it is not built on a replica of the fundamental,[58] the just minor third has been rejected, and with it the minor triad 10:12:15, by some authors who have accepted the harmonic usage of the just major third. Brandsma finds 32:27 better than 6:5 for harmonic as well as melodic use, calling 6:5 a horrible discord (unerträglichen Missklang), while he finds a diminished seventh best constructed as three superimposed 32:27 intervals.[59] Jonquière rejects the interval 6:5 in the construction of minor chords as too like the major (dur-ähnlich).[60]

The interval 6:5, whose first use is attributed by Kornerup to Erastothenes of Cyrene in about 200 B.C.,[61] has received substantial support in its own light. Ariel, in particular, likes this interval and regards it as the fundamental interval for a higher musical order (Grundintervall höherer Ordnung).[62]

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[58]Descartes

[59]Brandsma, op. cit., pp. 359-60.

[60]Jonquière, op. cit., p. 139, p. 145. On page 145, Jonquière points out that the just minor third calls into play a difference tone a perfect fifth below its upper member, a tone totally unsuitable if the minor third is the bottom part of a minor triad. Jonquiere does not choose to discuss the completely unrelated difference tones of the equal-tempered or Pythagorean minor thirds, however.

[61]Kornerup, Musical Acoustics based on a Pure Third System, columns 18-19.

[62]Ariel, op. cit., p. 74. For an elaboration of this point see chapter 10.

Harry Partch does not go so far as Ariel in singling out the interval 6:5, but he calls it "good to the ear,"[63] adding that its Pythagorean equivalent, 32:27, is "beyond the ear's capacity to determine accurately."[64] At first glance this indictment hardly seems relevant, since easiness to determine accurately does not in itself appear to be a satisfactory way to determine the consonance of an interval. But the implications of Partch's allegations go deeper. It is generally accepted that current tuning practice grants far greater leeway to variations in the third than in the octave or fifth.[65] But is not the ear's very willingness to accept a considerable alteration in the third a sign that it has not accepted the 300-cent minor third or the 400- or 408-cent major third as an acoustical standard? And does not the comparison between the relative stability of the small-number consonances (fifths, fourths, octaves) and the relative instability of the artificial consonances (the thirds and sixths of current use) at least suggest that when a musical society tunes its consonant intervals

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[63] Partch, _op_. _cit_., p. 241.

[64] _Ibid_., p. 278.

[65] "Octaves, fifths, and fourths present a certain problem of intonation, since the slightest deviation from the correct pitch in either note is more apparent to the ear than it would be in such intervals as the 6th and 3rd, where the mathematical ratio between the notes is more complex." -- Kent Kennan, discussing violin double-stops in _The Technique of Orchestration_. Indeed, the mathematical ratio of the thirds and sixths of common practice is "complex!" Small wonder that few musicians would refuse to concede Kennan his point on the flexibility of their intonation.

to small-number ratios it has a more stable basis for its intonation?

THE WEB OF FIFTHS AND THIRDS

With the acceptance of the 3rd and 5th partials as basic to most musical systems, and before proceeding to the consideration of the 7th partial (which remains controversial even among the advocates of multiple division), it is necessary to consider a body of acoustical data and speculation concerning the relationship between the 3rd and 5th partials, and the use of the intervals created by this relationship in the building of musical systems. Jose Würschmidt uses the term "Quinten-Terzengewebe" to describe what he regards as the total musical field for any system.[66] Wesley S. B. Woolhouse, an important pioneer writer in the field of multiple division, writing in 1835, claims that "Every interval may be determined by combinations of the Major 3rd, the Perfect 5th, and the Octave."[67]

Bosanquet asserts, "To provide a material of notes for musical performance it is theoretically requisite in the first place that to every note used we should possess

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[66]Würschmidt, Jose, Die rationellen Tonsysteme in Quinten-Terzengewebe, Zeitschrift für Physik, Bd. 46, Jan. 1928, p. 527.

[67]Woolhouse, W. S. B., Essay on Musical Intervals, Harmonics, and the Temperament of the Musical Scale, p. 73.

octave, fifth, and major third, up and down. Each of these being a note used we may require the same accessories to each, and so on."[68] Bosanquet continues, "If however we provide all the notes necessary for an extended system . . . we have endless series of fifths running up and down, and endless series of thirds running horizontally; and it is possible to show that no two of the notes will ever be exactly the same in pitch. Consequently in practice various approximations are employed, so as to reduce the number of notes required."[69]

As the just fifth and third are fixed intervals, the various relationships within the web of fifths and thirds are immutable. The most important of these relationships with their traditional terms and qualities are shown in Example 5.

Among the theorists who have dealt with multiple division in the past, a majority have built their systems to capitalize on features of the web of fifths and thirds. The "approximations" which Bosanquet notes above have led to two types of systems: equal-tempered systems in which every member can serve as third or fifth above or below another member; and unequal systems where for a greater measure of accuracy or simplicity or both, the complete mutuality of relationship is sacrificed and tones on the

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[68] Bosanquet, op. cit., p. 1.

[69] Ibid., p. 3.

Example 5: Relationships in Web of Fifths and Thirds[70]

Designation	Name	Ratio	Description
A.	Perfect Fifth	3:2	
B.	Major Third	5:4	
C.	Minor Third	6:5	A - B
D.	Major tone	9:8	Two fifths less one octave
E.	Minor tone	10:9	B - D
F.	Diatonic semitone	16:15	(A + B) deducted from one actave
G.	Major limma	27:25	3A - (2B + one octave)
H.	Chromatic semitone	25:24	D - G; E - F; 2B - A
I.	Minor limma	135:128	3A + B - two octaves; D - F
J.	Pythagorean limma	256:243	three octaves - 5A
K.	Major diesis[71]	128:125	one octave - 3B; G - I
L.	Minor diesis	3125:3072	5B - (one octave + A); H - K
M.	Syntonic comma	81:80	4A - (B + two octaves); D - E
N.	Minor comma	2048:2025	three octaves - (4A + 2B); K - M
O.	Pythagorean comma	531441:524288	12A - seven octaves
P.	Schisma	32805:32768	8A B - five octaves; M - N

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[70]Most of the materials in this chart are taken from a similar chart by Friedrich Opelt from his Allgemeine Theorie der Musik, p. 21.

[71]The Harvard Dictionary lists the major diesis as 135:128 (four minor thirds minus an octave), and calls the interval shown here, 128:125, a minor diesis.

fringe of the system lack fifth and/or third, above or below.

Würschmidt, in his theory of rational tone-systems, has made one of the most penetrating analyses of the qualities and possibilities of the various equal-tempered systems within the context of the web of fifths and thirds. He is particularly concerned with two kinds of minute intervals which he considers particularly important to tempered systems. The first kind he calls defining (definierend) intervals. These are the very small intervals in the web of 5ths and 3rds which are made to disappear (made equal to zero) in the process of building a specific tempered system. The syntonic comma, for example, is a defining interval in 12-tone temperament, as is the major diesis. The second kind, Würschmidt calls constructing (konstruierend) intervals. These are the intervals each of which is equal to one unit of a tempered system. The constructing intervals in 12-tone temperament are the various kinds of semitones such as 16:15 and 25:24. It is Würschmidt's basic contention that a rational system is one in which the largest defining interval is smaller than the smallest constructing interval.[72] The rational systems, according to Würschmidt

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[72] Würschmidt, Die rationellen Tonsystems, op. cit., p. 527. Würschmidt considers the syntonic comma, 81:80 and the major diesis, 128:125, to be the defining intervals in 12-tone temperament. The sizes of these intervals are 21.5 and 41.4 cents respectively. As constructing intervals, Würschmidt selects 16:15, 25:24, and 135:128, the smallest of which is 70 cents. 12-tone temperament is therefore

are 5-, 7-, 10-, 12-, 19-, 22-, 31-, 34-, 53-, and 118-tone temperaments.[73]

One of the great advantages of the concept of a web of fifths and thirds is that it provides a method of charting musical systems according to the intervals from which they are derived, thus making visual and easy to perceive the characteristic features of the system.

Bosanquet makes such a graph of the 12-tone system, which he calls the duodene, with the fifths vertical and the thirds horizontal.

Example 6: Bosanquet's Duodene.[74]

B^b	D	F#
E^b	G	B
A^b	C	E
D^b	F	A

rational. In choosing 135:128 as his third constructing interval Würschmidt is making a most unusual selection. The usual 12-tone system is built as follows, $(25:24)^5 \times (16:15)^4 \times (27:25)^3 = 2$. Würschmidt instead builds his 12-tone system as follows, $(16:15)^7 \times (135:128)^3 \times (25:24)^2 = 2$.

[73] Ibid., p. 556. Würschmidt does pose a limit on the expanse of his system for, if he did not, other small intervals far removed from musical practice would interfere with his calculations. He chooses as limits 11 fifths and 27 thirds in order to eliminate the very small discrepancies from the octave created by the 12th fifth (the Pythagorean comma) and by the 28th third (which exceeds the ninth octave by only a few cents).

[74] Bosanquet, op. cit., p. 2.

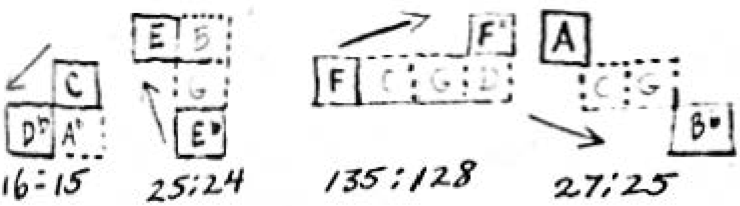
The usual practice in graphing fifths and thirds is to represent the fifths horizontally and the thirds vertically. Ex. 7 shows the same duodene as charted by Adriaan Fokker, who attributes it to the 17th century mathematician, Leonhard Euler.[75]

Example 7: Euler's Genus Diatonico-Chromaticum

A	E	B	F#
F	C	G	D
D^b	A^b	E^b	B^b

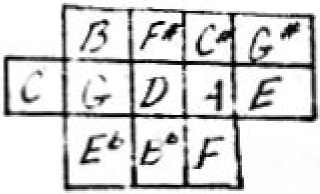
Wärschmidt, noting that the duodene requires four constructing intervals, chooses a different arrangement of tones. Example 8 shows the four constructing intervals in 12-tone temperament as they appear in Euler's system, while Example 9 shows Wärschmidt's solution, eliminating the interval 27:25.

Example 8: The Constructing Intervals in 12-tone Temperament



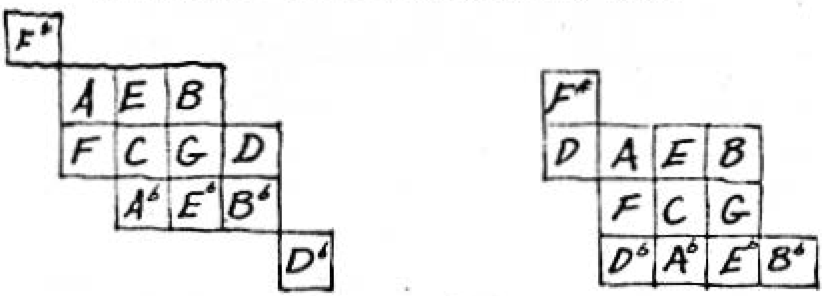
[75] Fokker, A. D., Just Intonation and the Combination of Harmonic Diatonic Melodic Groups, p. 9.

Example 9: Würschmidt's "normalized" 12-tone system.



Ariel, who was a contemporary of Würschmidt's and whose work was well known by him, was probably influenced by Würschmidt as well. In 1925, after Würschmidt had already developed his theories of the Quinten-Terzengewebe, but before he codified the 12-tone system as shown above, Ariel published his own views on the 12-tone system, using only the traditional three constructing intervals, 16:15, 25:24, and 27:25. Ariel offers two patterns in interval charts which, when graphed, turn out to be as is shown in Example 10.

Example 10: Two 12-tone systems by Ariel.[76]



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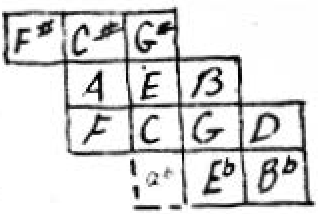
[76]Derived from the charts in Ariel, op. cit., p. 115.

It appears that Kornerup would agree with the second of the Ariel systems, for his own chart includes every one of Ariel's tones except F#, which Kornerup would probably select in the same manner.[77] One other pattern, somewhat similar to Ariel's is offered below in Example 11. It likewise employs only the same three constructing intervals. However, in spite of the strong preference shown by Ariel and Würschmidt for systems involving only three constructing intervals, it is hard to see why any of these systems is preferable to the Bosanquet duodene with its compact pattern involving four different sizes of semitone.

Example 11: An Additional 12-tone System



To Würschmidt, multiple division follows from the logical wish to eliminate the larger of the defining intervals. Since the deisis is the largest of the defining intervals in 12-tone temperament, the next rational system must provide separate tones for pitches separated by that interval. The diesis thus becomes the smallest of the

[77]Derived from Musical Acoustics Based on a Pure Third System, columns 27-8.

constructing intervals of the next rational system, which is 19-tone temperament. Used instead as a defining interval in 19-tone temperament is the 29.5-cent minor diesis, 3125:3072, consisting of the difference between the chromatic semitone 25:24 and the major diesis 128:125. Both of these latter intervals are constructing intervals, and it follows that since all constructing intervals are equal to one unit of a given system, the difference between any pair of constructing intervals must equal a defining interval. Therefore, in a rational system, the largest constructing interval must always be less than twice the size of the smallest constructing interval. Other of Würschmidt's rational systems will be examined in the chapters on the various specific systems of multiple division.

Just Systems: To theorists preferring exactitude of fifths and thirds to approximation, the problem has been to find ways of closing the open ends of their systems. To provide 12 central tones with Bosanquet's minimal requirement (a fifth and a third both above and below) would require 26 tones to the octave, the outer 14 of which would serve merely as auxiliaries without the acoustical resources to become tonics in their own right. Making potential tonics of all the tones in the traditional twelve major keys would require a minimum of 37 tones per octave, 18 of them lacking at least one auxiliary. Nevertheless, many theorists have tried to come to terms with the infinitude of possibilities, and just organs and harmonia have been built with almost every possible number of tones to the octave. Barbour examines a number of these instruments[78] beginning on page 108, and Partch studies several in great detail.

Within the web of fifths and thirds, perhaps the most remarkable near-identity is the schisma, which represents the difference between 8 fifths plus a third and five octaves: 32805:32768, or 1.945 cents. Even those theorists who have been implacably opposed to substituting the approximations inherent in equal temperaments for just ratios have tended to accept the principle of an alteration of each fifth by 1/8 schisma (about a quarter of a cent) in order that every eighth fifth might produce a just major third when deducted from five octaves. The treatment of the schisma as an identity can simplify a musical system considerably. Clark Jones rightly remarks that 15 major keys can, with the assistance of a minute temperament of 1/8 schisma, be realized with as few as 24 tones to the octave, whereas 35 would be required if the schisma were not treated as an identity.[79]

The Norwegian composer Eivind Groven also uses

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[78] In Tuning and Temperament.

[79] Jones, Remarks on Just Intonation and Musical Scales, Acoustical Society Journal, 1947, p. 727.

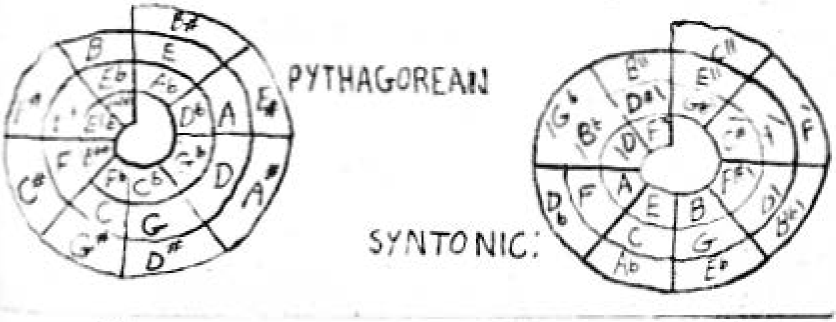
schismatic temperament for his just organ,[80] but feels compelled to add 12 tones to the 24 recommended by Jones, bringing to 36 the number of tones to the octave.

The schisma is the subject of a classical approximation, possibly mislabeled a theorem, Helmholtz' Theorem. This states that the schisma, produced as outlined above, is approximately equal to the error of the fifth in 12-tone temperament. The respective figures in cents are:

Schisma: 1.945 cents
Error of 5th: 1.955 cents.

The use of the schisma as an identity converts the open-ended plane graph into a kind of spiral, or partly detached ring. The perimeter represents a cycle of fifths while the spokes represent a series of thirds. Example 12 illustrates the system of 24-tones advocated by Jones. Two possible nomenclatures are shown.

Example 12: 24-Tone Schismatic Temperament



[80]Groven, My Untempered Organ, Organ Institute Quarterly, Vol 5:3, Summer 1955.

Eivind Groven extends the spiral by 540 degrees, and uses just such a graph to illustrate his article which is cited above.

The Schisma in Pythagorean systems: Since 8 fifths deducted from 5 octaves leaves a major third in schismatic temperaments, a system of fifths yields a just third only if there are at least 8 superimposed fifths. After the first 8 tones, each additional tone of a Pythagorean system will yield the illusion of a just third. Barbour and Kuttner point out that this gives the distant keys of Pythagorean tuning a close resemblance to the near keys in just intonation, while Kornerup, who strongly disapproves of Pythagorean tuning,[81] says of the phenomenon of almost-just thirds over the "back fifths" of the Pythagorean system, "It is the playful way of nature that all exaggeration corrects itself."[82]

Because of the relatively close correspondence between twelve fifths and seven octaves, the tones of an extended cycle of pure (or sub-schismatically tempered) fifths tend to pile up in layers of twelve, each removed from the other by about a comma . . . a Pythagorean comma in the case of pure fifths, and an interval slightly smaller than a

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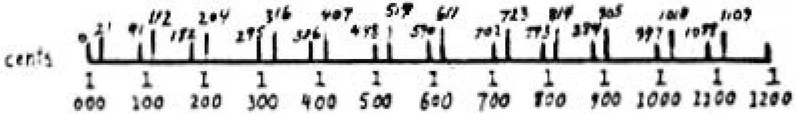
[81] Kornerup likens the octave, fifth, and third, to the three primary colors, maintaining that Pythagoreanism is the musical equivalent of color-blindedness. Musical Acoustics Based on the Pure Third System, column 29.

[82] Kornerup, Acoustic Methods at Work, 1934, p. 19.

syntonic comma in the case of sub-schismatically altered fifths. The 24 tones of Jones' spiral are grouped as follows when converted into a pitch continuum.

Example 13: Pitch Continuum of the 24-Tone Pythagorean Spiral (slightly altered by schismatic temperament)



With each addition of 12 tones another "layer" is added, one comma higher than the preceding one. This fact possibly explains why proponents of this kind of system have tended to adopt systems whose tones numbered multiples of twelve. It will be seen in example 13, above, that the intervals between each of the twelve close groups are of two different sizes, the one being approximately 70 cents, the other, 91 cents. The smaller of these two intervals is approximately three commas in size, while the larger one is about four. In the 24-tone system shown in example 13, there are seven 3-comma gaps, and five 4-comma gaps. As many theorists have pointed out, it is possible to bring the web of fifths and thirds to a very close approximation of a closed system by continuing the spiral until these comma gaps are filled in. A total of 29 additional tones are needed to fill these gaps which brings to 53 the total number of tones in the system. A 53-tone Pythagorean system is almost identical with 53-tone equal temperament, as 53 fifths (37203.35 cents) is nearly precisely equal to 31 octaves (37200.00 cents). A single fifth in 53-tone temperament is distinguished from the just fifth by only 0.06 of a cent.[83]

The building of 12-tone layers separated by a comma encouraged theorists to use traditional tonal nomenclature, with the addition of a dash over or under the note-name to indicate raising or lowering by a comma. This practice, according to Kornerup, evolved from a system of comma-differentiation by upper and lower case letters used by Hauptmann in 1853 to the fully developed plan of Oettingen in 1866, in which dashes above the note-names signified lowering by a comma. Helmholtz, according to Kornerup, duplicated this system in 1870, but with the dashes used below the letters to lower the pitch. Eitz in 1891 still used the system, but with addition and division signs after the note-names rather than dashes.[84]

Oettingen, Helmholtz and Eitz all used Pythagorean nomenclature, wherein E, representing the just third above

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[83] The size of the fifth in 53-tone temperament as well as much of the data about interval sizes throughout this paper is gathered from an impressive though unpublished collection of charts appended to An Introduction to Tuning and Temperament, by Charles S. Kent.

[84] Kornerup, Musical Acoustics based on a Pure Third System, columns 21-22.

C, appears with the sign of comma modification. The G# 25:16 appears with two comma signs indicating that it is two commas removed from the G# of the Pythagorean series. Kornerup (1922) offers a chart showing the nomenclature of his predecessors for four tones as derived from C as center. He also shows his own nomenclature, which is based on a just third system wherein it is the Pythagorean intervals, rather than the syntonic intervals, which are altered by comma signs. Example 14 shows excerpts from Kornerup's chart.

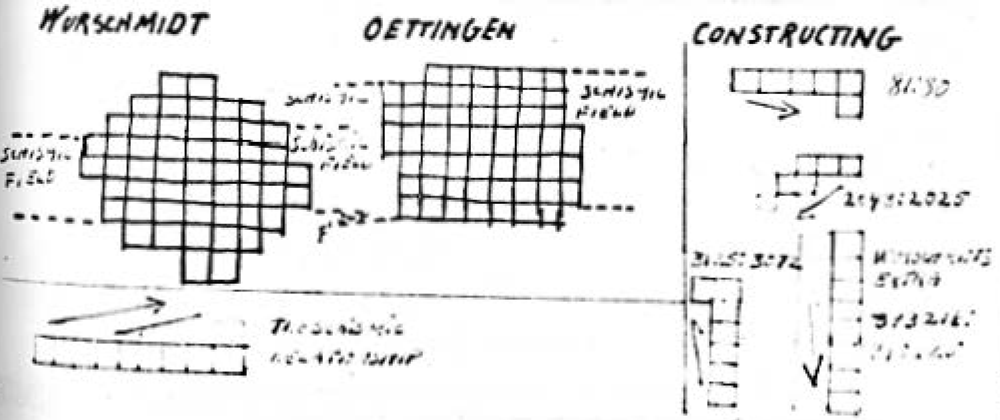
Example 14:

Ratio	Cents	Oettingen	Helmholtz (1870 version)	Eitz	Kornerup	Pythagorean (not in Kornerup's chart)
25:16	-773	$\overline{\overline{g\#}}$	$\underline{\underline{g\#}}$	g# ÷2	g#	Bbbb
5:4	386	$\overline{e}$	$\underline{e}$	e ÷1	e	F♭
10:9	182	$\overline{d}$	$\underline{d}$	d ÷1	d	E♭♭
9:8	204	d	d	d	d+	D

The debate between adherents of the two nomenclatures has been fierce. Würschmidt, an adherent of syntonic nomenclature, and J. Wallot, a disciple of Oettingen, engaged in a bitter debate in the pages of the Zeitschrift für Physik[85] in 1921 and 1922.

Würschmidt compares his 53-tone system with Oettingen's in diagram form, attempting further to confound Wallot. Note, however, in Example 15, that Oettingen's is more compact than Würschmidt's and that it makes more extensive use of the schismic relation (shown by the dotted-line extensions in Example 15).

Example 15: Würschmidt's and Oettingen's 53-tone Systems with Defining and Constructing Intervals shown.



It should be noted that a truly Pythagorean 53-tone system would appear on a diagram as a single horizontal row of 53 fifths. Not even Oettingen's nomenclature,

[85]See particularly Wallot in volume 4, p. 157, and Würschmidt in volume 5, page 111. Würschmidt ascribes the syntonic nomenclature he uses to H. Starke, author of Physikalische Musiklehre (of which Würschmidt cites pages 99 and 100). Würschmidt claims as the chief advantage to syntonic nomenclature that many fewer comma-alterations are needed than in Oettingen's system, a claim which is justified.

complicated by comma-signs, employs the multiple sharps and flats that would be necessary to reflect such a diagram accurately. It would appear that while Würschmidt's nomenclature does commend itself by its relative simplicity, Oettingen's system, as shown in Example 15, above, is the superior one, even in Würschmidt's own terms: It is the more compact; it is the one which uses only three constructing intervals; it makes much greater use of the schisma as an identity.

The two basic nomenclatures of just musical systems and of 53-tone temperament are reflected by two distinct divisions of the 53-tone system to produce the diatonic scale. Joachim Steiner, advocating 53-tone temperament as a basis for measurement refers to the Pythagorean division 9 9 4 9 9 9 4 as string-lydian and to the just division 9 8 5 9 8 9 5 as flute-lydian.[86] The dilemma of the alternate scales and nomenclatures results from the conversion of the syntonic comma from defining to constructing interval. Our note-names are based on the assumption, valid in 12-tone temperament, that two pitches separated by a syntonic comma are in reality one and the same.

The web of 5ths and 3rds does not end with 53 tones. Würschmidt has gone on to diagram 118-tone temperament. The web is, indeed, endless, and it would appear advisable to

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[86] Einfluss der reinen Stimmung auf die Entwicklung der Musik, Bericht von III. Kongress der Internationalen Musikgesellschaft, page 387.

extricate ourselves and move on to the 7th partial.

THE SEVENTH PARTIAL: 7:4, 7:5, 7:6 AND THEIR INVERSIONS

> "In the flora and fauna the figures 1, 2, 3, and 5 are ruling according to the law of precedence that 1, 2, and 3 (as well as multiplies of these figures with each other) are ruling on a lower stage of development; for instance in the flowers of monocotyledonous plants and with zoöphytes; while 5 (and multiples as 10, 15, and so forth) indicate a higher phase of development, -- in the flowers of bicotyledonus plants and with echimoderms (starfish, crinoideans, arctiniae), while other prime numbers indicate teratologies."[87]

The above categorical opposition to the role of the number 7 appears in a theoretical treatise on music. It is not, as one might think, from an opponent of the expansion of musical resources. The statement is by Kornerup, a 20th century advocate of multiple division. It would appear that in whatever tonal worlds may develop, the 7th partial will remain the subject of controversy.

Kornerup, in his other writings, acknowledges the seventh partial. He admits its place among some of the musical systems of the past and present,[88] and even provides an approximation for the interval 7:6 in his Golden

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[87] Kornerup, Musical Acoustics based on a Pure Third System, column 31.

[88] He asserts that the arithmetical division 10:10, 10:9, 10:8, 10:7, 10:6, 10:5 is the basic type for oriental pentatonic music. Acoustics Methods of Work, 1934, p. 11. This work is ten years more recent than the one in which he attacks the 7th partial so vehemently.

System.[89] More implacably opposed to the inclusion of the 7th partial in any kind of musical system is Ariel. He distinguishes between the "natural" number series (which includes all numbers) and the "harmonic" series (which includes no primes or products of primes higher than five).[90] Ariel's opposition touches every aspect of the seventh partial, and is categorical and absolute.

Danielou projects the view that the seventh partial is beyond the scope of normal human perception,[91] and that its introduction would therefore be "dangerous."

In their opposition to the incorporation of the 7th partial into any new musical systems, these men follow a long and distinguished line of theorists whose ranks include Zarlino, Rameau, Oettingen, Riemann, and Hindemith.[92] In his definitive work on the 7th partial,[93] Martin Vogel

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[89]For a thorough discussion of Kornerup's Golden System, see Chapter 9. Kornerup regards 7:6 as smaller than 6:5 by a diesis which, in the golden system, is .618 of a small semitone. The golden approximation of 7:6 is one of the best in Kornerup's system, 265.93 cents, within one cent of the natural ratio (which is 266.87 cents). Kornerup's representation of 7:4 is much less accurate, however.

[90]Ariel, op. cit., p. 47. Ariel goes on to point out (p. 51) "Wenn..zwei Sänger einen Zweiklang singen, dessen Töne genau im Verhältnis 6:7 schwingen, dann ist das weder ein musikalisches Intervall noch die Temperierung eines solchen; es ist überhaupt keine Musik." (If two singers sing at an interval 7:6 apart, the result is neither a musical interval nor the temperament of such an interval; it is indeed not music at all).

[91]Danielou, op. cit., p. 231.

[92]These names are cited by Partch, op. cit., p. 93.

[93]Zahl 7 in der Spekulativen Musiktheorie. Vogel is Privatdozent at the University of Bonn.

leaves little doubt that every conceivable argument which present-day senarists might wish to use has already been offered. However, as Vogel shows, much of the opposition to the theoretical endorsement of the 7th partial has been based primarily on its unavailability in present-day tuning and in its exclusion from the diatonic scale.[94]

There is a large measure of ambivalence toward the seventh partial on the part of many advocates of multiple division. Friedrich Opelt, within a single text,[95] presents a "complete" table of basic intervals from which the seventh partial is completely absent,[96] a table of consonances which includes ratios of the 7th partial,[97] a statement that the seventh partial is disturbing (störend),[98] the statement[99] that all consonances are derived from the single chord 4:5:6:7:8, the recognition of 31-tone temperament as a means of including the seventh partial within a musical system, and finally the decision to exclude the seventh partial from further musical speculation because of the

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[94]Vogel cites the theorist Chladni, who, in referring to Kirnberger's name for the tone created by the 7th partial from C (I), cracked that the natural seventh was named i after the imperative form of the Latin _ire_, to go, and meant "Go away; we have no place for you here."

[95]_Allgemeine Theorie der Musik_, 18.

[96]p. 21.

[97]p. 30.

[98]p. 65.

[99]p. 66.

technical difficulties 31-tone temperament would entail.[100]

Euler, the 17th century mathematician on whose theories of the Greek genera a major current school of 31-tone tempered music has been founded, demonstrates much of the ambivalence toward the seventh partial shown more recently by Opelt. Beginning with speculations about mode as based on the senario, Euler rejects acknowledged Greek tunings based on septimal intervals as imperfect and harsh.[101] Speculating about their use nonetheless, Euler becomes increasingly interested in them and, later, endorses the use of the seventh partial. He evolves a mathematical formula for measuring relative degrees of consonance (by adding factors and exponents) whereby the natural minor seventh is the most consonant of the minor sevenths whenever the interval is increased by the interposition of one or more octaves.[102]

Support for the inclusion of the 7th partial in musical theory has as long and distinguished a history as has its opposition. Among the figures who have insisted that a role be found for the number 7 have been Mersenne,

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[100] p. 67.

[101] Euler's Tentament Novae Theoriae, unpubl. doctoral dissertation by Charles S. Smith, p. 175.

[102] Euler's ingenious though highly questionable system for measuring consonance is treated at some length in Vogel, op. cit., esp. p. 78.

Tartini, Fétis, Serre, Hauptmann,[103] and Kirnberger.[104] The well-known critic, Donald Francis Tovey, is at the end of a long line of musicians of many generations when he complains that music as it is now constituted "contains no room for so simple a thing as the 7th partial."[105]

Many proponents of multiple division have been interested in the possible role of the 7th partial in enlarged musical systems. Yasser, who calls the 7th partial "flavorless" within the diatonic system,[106] incorporates it nonetheless into the body of consonant sonorities of his "supra-diatonic" system.[107]

The interest in the seventh partial on the part of some theorists has been so keen that there have even been proposed "just" 12-tone systems whose non-diatonic members are to be drawn from septimal ratios. One such proposal is by A. M. Awraamoff in the first volume of Melos, 1920. He offers a septimal tuning for the 12 keys of the piano in the name of "freeing" music from the shackles of equal

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[103] Partch, op. cit., p. 93.

[104] Kirnberger's interest in the 7th partial is treated at some length in Vogel, op. cit.

[105] Encyclopaedia Brittanica, 11th Edition, quoted by Fokker, Expériences Musicales, p. 133.

[106] A Theory of Evolving Tonality, footnote to page 193.

[107] For details on Yasser's supra-diatonic system and its use of the 7th partial, see below, Chapter 11.

temperament.[108] Awraamoff's position appears inconsistent, however, for while his theory postulates a just, non-septimally derived diatonic scale with septimal tones between scale degrees, the actual pitches of his system yield no such diatonic scale. The harmonic minor of F is, however, possible in the scale which is shown in Example 16.

Example 16: Awraamoff's Septimal "Just" Scale

Intervals between successive tones:		9/8		64/63		21/20		25/24		21/20		64/63		9/8		16/15		15/14		49/48		21/20		16/15	
Pitch names given by Awraamoff:	F		G		Y		Ab		A		B		Bb		C		Db		Δ		Ɛ		E		F
Intervals upwards from F:	1		9/8		8/7		6/5		5/4		21/16		4/3		3/2		8/5		12/7		7/4		15/8		2/1

The weaknesses of this scale are evident. The intervals vary in size from 9:8 to 64:63 and include seven different sizes. Absolutely no scale patterns and very few harmonic patterns are transposable at all, and these only to a few pitches within the system. Awraamoff's concept of tonality is evidently quite static. With the exception of the interval between Ab and A (25:24) and the interval between Δ and Ɛ (49:48), the intervals of Awraamoff's

108 Awraamoff, A. W., Jenseits von Temperierung and Tonalität, Melos I, p. 131. Awraamoff's system has the dubious distinction of having been called worse than "the poorest tuning system shown in this book" by Barbour, Tuning and Temperament, p. 24.

scale would fit fairly comfortably into 41-tone equal temperament. Exactly as it stands, however, Awraamoff's entire scale could not be approximated by any temperament within the range of this study.

Adriaan Fokker has recently produced a 12-tone genus which achieves what Awraamoff sought to obtain. Fokker combines four septimal tetrads 4:5:6:7, the first two built harmonically on F and C, the other two built ornamentally downward from F# and B.[109] The sixteen tones thereby obtained include four identities, thereby reducing the number of distinct pitches to twelve. The result involves much less dislocation in the tones of present tuning practice than does Awraamoff's system. The five constructing intervals range in size from 28:27 to 15:14, involving a range of less than 2 to 1 and only slightly greater than the range of constructing intervals in the traditional just 12-tone scale. The entire system fits reasonably well into 31-tone temperament with the single intervals representing 2 or 3 units, generally somewhat extended or shortened. Fokker's 12-tone septimal scale is shown in Example 17.

It should be emphasized that Fokker's septimal 12-tone scale is offered as one of many possibilities, almost as a curiosity, while Awraamoff's curiosity is suggested in all seriousness.

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[109] Fokker, Just Intonation, p. 40.

Example 17: Fokker's Diatonic Genus with 4 Added Sevenths

Intervals between successive tones:	15/14	21/20	28/27	15/14	16/15	135/128	16/15	15/14	28/27	21/20	15/14	16/15	
Pitch names suggested by Fokker's signs:	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
Intervals upwards from C:	1	15/14	9/8	7/6	5/4	4/3	45/32	3/2	45/28	5/3	7/4	15/8	2

A system somewhat similar to Awraamoff's and Fokker's is offered by the Englishman Perrett.[110] His system rightfully belongs among the proposals for multiple division which are considered in succeeding chapters (see Chapter 7). The basis for Perrett's system is the construction of two diatonic scales a septimal semitone (21:20) apart. It is quite understandable that theorists who are interested in the 7th partial are unlikely to be satisfied with any 12-tone system, and Perrett's 14-tone basic scale (he later extends it to 19 tones) is but a first step away from 12-tone thinking under the influence of the 7th partial.

The various intervals created by the 7th partial have the following discrepancies in 12-tone temperament:

7:4	32 cents too large
7:5	18 cents too large
7:6	34 cents too large
9:7	36 cents too small

[110] Perrett, Some Questions of Musical Theory, 1926-. His scale is in the appendix to Vol. II.

Nonetheless, many theorists have held that the seventh partial explains at least some of the phenomena of the music of the current repertoire. The dominant seventh, in particular, has frequently been attributed to the partials 4, 5, 6, and 7. That many advocates of the expansion of musical resources should argue, as does Fokker, that 4:5:6:7 is the "primary chord" of music,[111] is hardly surprising. Even some of the theorists who have been loath to accept the validity of the 5th partial have accepted the 7th partial as a musical reality. A much-cited survey by Cornu and Mercadier late in the 19th century, which showed a preference for artificial thirds in place of the 5th partial except in the harmonic use of the major third, showed a general preference for the natural seventh in dominant harmony. More than that, Brandsma, who agreed with Cornu and Mercadier about the 5th partial, considers the interval 7:4 to be present in the rich harmonies of Wagner and responsible for their stunning ("glänzende") effect.[112]

Brandsma's thesis that the interval 7:4 pervades Wagner's harmonies leads to interesting speculation about the invertibility of the sonority 4:5:6:7. The smallest

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[111] Fokker, Just Intonation, p. 10.

[112] Brandsma, op. cit., p. 556. Further emphatic support of the role of the 7th partial in dominant harmony is offered by L. Norden in A New Study of Intervals, Organ Institute Quarterly, Vol 5:I, page 31.

rational integers which can be used to indicate the inversion of this sound are 60:70:84:105. Nevertheless, Fokker uses such sonorities, which he calls chords of the sub-seventh, as does Harry Partch.

The speculation arises from the use by Wagner of a vast number of half-diminished seventh chords. The half-diminished seventh, in traditional practice, is the exact inversion of the dominant seventh and is therefore to the chord of the sub-seventh precisely as the dominant seventh is to the chord 4:5:6:7. Brandsma never mentions the half-diminished seventh explicitly. Elsewhere in his article, he states that the diminished seventh should consist of three Pythagorean minor thirds, leaving one to wonder whether he considers the half-diminished seventh to be based on different lower intervals than the diminished seventh. It is possible, but hardly likely, that Brandsma meant the half-diminished chord to consist of two Pythagorean minor thirds with a natural seventh perched on top, 381 cents above the diminished fifth. Owing to his disinclination for the 5th partial, it is possible that the above would be his preference over the more logical $\frac{1}{7}:\frac{1}{6}:\frac{1}{5}:\frac{1}{4}$ or 60:70:84:105.

While the interval 7:4 has been singled out by some writers such as Brandsma, 7:5 and 7:6 have also drawn the particular interest of others. Perrett finds in the tritone 7:5 (583 cents) great beauty which he finds lacking in the equal-tempered substitute.[113] Perrett cites Tartini in support of his conviction that 7:5 is the proper ratio for the augmented fourth.

7:6 has been offered by Awraamoff, Fokker, and others, as the lower interval of an important septimal minor triad, 6:7:9. This sonority is offered by Awraamoff as the explanation for the supertonic chord in just intonation (supposedly thereby ending the riddle of its apparent dissonance). It is difficult to see the basis for Awraamoff's assertion. The chord 27:32:40, which is the chord of the second degree of the standard just major scale, bears no more of a resemblance of 6:7:9 than it does to 10:12:15, the standard just minor triad. While 32:27 is a comma smaller than 6:5, it is more than a comma larger than 7:6, and the septimal minor triad, like the traditional minor triad, contains a perfect fifth which, if used for supertonic harmony, must throw some other fifth out of tune.

The sonority 6:7:9, like the sonority 4:5:6:7, has been cited considerably more often than its inversion, leading one to question anew whether the sonorities extrapolated from the harmonic series are, in fact, invertible. Partch and Fokker repeatedly insist that they are, but,

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113 So enamored is Perrett of the sound of 7:5, that he postulates the invention of the enharmonic genus in ancient times to a chance discovery of this interval by Olympus, owing to a slight mistuning of his lyre. Some Questions of Musical Theory, p. 3.

outside of their pages, one is unlikely to find any reference to a septimal major, 14:18:21. In experiments related to the writing of this paper, the septimal minor chord was enthusiastically received while the septimal major was not. Those who are as argumentatively inclined as the many distinguished theorists who have debated in the past the invertibility of the 3rd and 5th partials will have a fine new bone to chew upon should the 7th partial assume a greater role in the music of the future.

<u>The Web of 5ths, 3rds, and 7ths</u>: The addition of the harmonic seventh to the web of thirds and fifths adds a new array of possible intervals which can be used in place of semitone, diesis, and comma. The proliferation of these intervals was regarded as a great threat by Ariel, and their absorption into a musical system does pose a considerable problem. Example 18 shows a few of the most prominent of these new intervals.

The four intervals which are bracketed in Example 18 are the ones Fokker uses as constructing intervals for his 31-tone system. The bottom one is one of his defining intervals, others of which include the syntonic comma, 21.5 cents, and the 8-cent interval, 1029:1024, consisting of three harmonic sevenths and a fifth, minus three octaves.[114]

The two near-identities (225:224 and 1029:1024) are

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[114] Fokker, *Les Mathématiques et la Musique*, p. 26.

Example 18: Septimal Intervals

Ratio	Size in cents	Description
15:14	118.4	Perfect fifth plus major third, minus natural seventh.
21:20	85	Perfect fifth plus natural seventh, minus octave and third.
28:27	63	Natural seventh minus a Pythagorean major sixth.
36:35	49	Two perfect fifths, minus a major third and natural seventh.
49:48	36	Two natural sevenths, minus an octave and a fifth.
50:49	35	An octave and two major thirds, minus two natural sevenths.
64:63	27	Two octaves, minus two fifths and one natural seventh.
225:224	7.7	Two fifths and two thirds, minus an octave and a natural seventh.

in a somewhat less perfect way to septimal just intonation what the schismatic relationship is to senary just intonation. But the 7th partial adds a third dimension to the web or tone-lattice as Fokker calls it. Because of the added difficulties inherent in this third dimension, nobody has as yet proposed a septimal semi-closed, "rounded" just system comparable with Jones' or Groven's senary just systems. Septimal schismatic temperaments do not, as yet, exist. There are, on the one hand, systems such as Perrett's and Partch's (see Chapter 7) which incorporate the 7th partial but make no attempt at complete or near-complete mutuality (transposability). On the other hand there is 31-tone temperament with its close approximation for the intervals of the 7th partial, as is shown in Example 19, below.

Example 19:

Errors of Septimal Intervals in 31-tone Temperament

Interval	Error
7:4	1.1 cents
7:5	2.1 cents
7:6	4.1 cents
9:7	9.3 cents

While no semi-closed just system specifically designed to include the seventh partial has been advanced, Bosanquet did not neglect to suggest methods of approximating the seventh partial within his senary systems. The standard minor seventh, 16:9, when flattened by a Pythagorean comma, becomes almost equal to the natural seventh.[115] It may be implied from this that the comma-layer systems of Helmholtz, Bosanquet, Eitz, Groven, and others can offer a close approximation of the 7th partial. However, Bosanquet's approximation is 4 cents in error, and another 2 cents must be added to the error when the more usual syntonic comma is substituted for the Pythagorean. The 6-cent approximation for the natural seventh in many of the systems of fifths and thirds may well be worth noting, although this might still

[115]Bosanquet, op. cit., p. 41.

be an excessive discrepancy for an interval as unfamiliar as the natural seventh.

Meantone temperament offered an exceptionally fine approximation of the natural seventh in the form of its augmented sixth. This is forcefully brought out by such early theorists as Kirnberger, and by Bosanquet[116] and Kornerup.[117]

The availability of the 7th partial as an overtone for brass instruments has been exploited occasionally as by Rossini,[118] Beethoven,[119] and more recently by Britten.[120] Bartok is cited[121] as having advocated its use, arguing that folk-practice and creative necessity justify its incorporation into musical theory.

The possibility of making the intervals of the 7th partial available at any scale degree in any musical context,

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116 Ibid., p. 42.

117 Kornerup, Die Vorläufer der gleichschwebenden Temperatur, pp. 1-2.

118 In a fanfare cited by Gevaert, Histoire et Théorie de La Musique de l'Antiquité, p. 315.

119 Trio of Eroica Symphony, third movement. It is of course played higher today, but the concert Db's for third horn are produced by the seventh partial and were probably played "low" since the usual practice was to lower more than to raise the natural pitch by hand. Furthermore, these Db's sound very beautiful when played "low."

120 In the Serenade for Tenor, Horn, and Strings, Prelude and Postlude.

121 By Fokker in La Gamme..., page 152. Article cited is from the Hungarian publication Uj Idok.

although rejected by some advocates of multiple division, is to others the prime reason for looking beyond the horizon of the 12-tone system.

THE HIGHER PARTIALS

In the expansion of the field of usable elements from the harmonic series, once the 7th partial is accepted, the 8th, 9th, and 10th partials follow without difficulty. The 8th and 10th partials are, of course, replicas of the fundamental and of the 5th partial, while the 9th partial is derived from two superimposed fifths, and is generally conceded to be the basis of the supertonic degree of the just scale.[122] The next controversial point in the expansion of the usable range in the harmonic series is the 11th partial and, as in the case of the 7th partial, the dispute has been fierce, if waged on a more limited scale.

The use of intervals based on ratios involving the number 11 has a long history. Students of various ancient and primitive cultures have taken measurements of pipes and discovered in places as diverse as Greece and Norway evidence of a preference for arithmetical division of the pipe

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[122] Certain theorists, such as Ariel and Kornerup, who tend to deny the musicality of the 7th partial as well, claim the supertonic to be 10:9 above the tonic rather than 9:8. There is still a 9 in the ratio, however, and at some point all "just" diatonic scales must contain two consecutive fifths, creating together the relationship that characterizes the 9th partial.

by twelfths of its length which produces the pentachord $\frac{1}{12}\cdot\frac{1}{11}\cdot\frac{1}{10}\cdot\frac{1}{9}\cdot\frac{1}{8}$. Erik Eggen, the Norwegian musicologist, considers this pentachord, as well as its inversion, 8:9:10:11:12, to be present in the musical scales of many peoples, including the Norwegians themselves.[123] Barbour lists a scale based on the tetrachord $\frac{1}{12}\cdot\frac{1}{11}\cdot\frac{1}{10}\cdot\frac{1}{9}$ as being among the Greek tunings listed by Ptolemy.[124] The interval 12:11 is also cited by Barbour as the middle interval of the tetrachords in Ptolemy's Chromatic Syntonon.[125]

According to Partch, the Arabian lute-player Zalzal recognized the 11th partial as an element in his music, and the Renaissance theorist Vicentino accepted the 11th partial in theory if not in practice.[126] A number of writers have cited the 11th partial as the basis for quarter-tone music (24-tone equal temperament) owing to the close relationship between the intervals 12:11 (150.6 cents), 11:9 (347.4 cents) and 11:8 (551.3 cents) and the corresponding intervals in 24-tone temperament (150, 350, and 550 cents respectively). Partch and Eggen point out

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[123] Eggen, Erik, Zur Entstehung und Entwicklung der Skala, pp. 104-5. Eggen notes a high 4th in Norwegian folk-songs and considers this tone to represent the 11th partial in the diapente 8:9:10:11:12. He elaborates on this in his Skala-Studier.

[124] The Diatonic Hemiolon. Tuning and Temperament, p. 21.

[125] Ibid., p. 18.

[126] Op. cit., p. 251.

this relationship, but perhaps more significant is the affirmation of the role of the 11th partial in quarter-tone music by Ivan Wyschnegradsky, one of the most ardent of the quarter-tonists. In his _Manuel d'Harmonie À Quarts de Ton_, 1933, Wyschnegradsky cites the 11th partial as the best basis for 24-tone temperament.[127] He suggests further, however, that it is also possible to regard the 7th partial as responsible for 19/24 (a discrepancy of 18 cents), the 13th for 17/24, and the 29th through 41st partials as the basis for the complete scale from 21/24 through the next 9/24.

Wyschnegradsky's rather free use of what are probably unperceivable relationships suggests that he is merely looking for an acoustical basis for a pre-existing musical system. Partch, however, creates his musical system out of a determination, _a priori_, to include the 11th partial. Partch makes some effort to round out his system by adding tones to fill the largest intervals, but he does not try to approximate a closed tempered system, although he takes the trouble to calculate that the harmonic series through 11 can be reasonably approximated by a temperament of 113 tones to the octave.[128] Such a number is too high even for Partch.

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[127] Wyschnegradsky, _op. cit._, p. 23.

[128] Partch, _op. cit._, p. 307. For a critical evaluation of 113-tone temperament, see chapter 6.

Lindsay Norden speculates that the 11th partial may well be the next to be brought into the body of music, although he does not consider it to have been of any importance in the art of harmony up till now.[129] Schönberg, on the other hand, regards the 11th partial, as well as the 13th, as basic to the chromatic scale.[130] Considering Schönberg's known views on the necessity of overcoming traditional ways of hearing triadic relationships, the theory enunciated in his article, <u>Problems of Harmony</u>, relating all tones in the 12-tone system to the partials of I, IV, and V, is quite astounding.

Example 20: Schönberg's Acoustical Source

Partial:	1	2	3	4	5	6	7	8	9	10	11	12	13
	C	C	G	C	E	G	Bb	CC	D	E	F#	G	Ab
	F	F	C	F	A	C	Eb	F	G	A	B	C	Db
	G	G	D	G	B	D	F	G	A	B	C#	D	Eb

Schönberg's espousal of partials up to the 13th as being present in 12-tone temperament is attacked quite vehemently by Yasser.[131] Yasser points out that the C# derived as the 11th partial of G represents a pitch almost a semitone from the Db representing the 13th partial of F,

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129 <u>A New Study of Intervals</u>, p. 38.

130 Schönberg, <u>Problems of Harmony</u>, Modern Music XI, p. 170.

131 Yasser, <u>A Letter from Arnold Schönberg</u>, AMS Journal, 1953.

yet both are represented by the same pitch in equal temperament. In spite of discrepancies of nearly 50 cents, Schönberg reaffirms his preference for equal temperament, likening natural intonation to nudity and temperament to presentable dress . . . possibly the most evocative defense of equal temperament that ever has been attempted.[132]

If Schönberg pays a possibly excessive obeisance to the upper partials through 13 in his concept of the 12-tone system, Yasser does the same with respect to the same partials as the basis for the consonant units of his 19-tone system. The errors of the intervals derived from some of these partials sometimes exceed 20 cents and approach the limit of possible error in 19-tone temperament (about 31.6 cents). Examples of such errors which Yasser himself has calculated[133] include 13:8, 22 cents, 11:7, 24 cents, 12:11, 24 cents.

Since the harmonic series is endless, any finite musical system must draw its limits. The Pythagoreans allow no primes higher than 3. Many have been found who draw the line after 5, or 7. Partch, claiming his reasons to be practical and arbitrary rather than based on an immutable theoretical concept, stops after 11. Yasser and Schönberg build on very loose approximations of the

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[132] Yasser, Ibid., p. 55.

[133] Unpublished letter from Joseph Yasser to John Redfield, May 9, 1933. His calculations, in centitones, have here been converted into cents.

partials through 13. One rather exceptional group of English theorists, starting with MacFarren, have held the 17th partial to be an important musical element[134] even while rejecting the 11th and 13th. They may have been influenced largely by the close approximation of the 17th partial in 12-tone equal temperament. Perrett, however, who also espouses the 17th partial[135] while rejecting both the 11th and the 13th, is a sharp critic of 12-tone temperament and an advocate of multiple division. He calculates that with the primes 3, 5, 7, and 17, 171-tone temperament is the nearly perfect scale (see chapter 6).

A possible weakness in any musical theory involving accurate renditions of the upper partials as an objective, lies in newly found data showing the overtones of actual sounding bodies to represent their theoretical pitches with increasing inaccuracy as their distance from the fundamental increases. The noted acoustician, Robert W. Young, has dealt with this phenomenon extensively,[136] and it is possible that further exploration may bring into question many acoustical arguments for just systems involving the higher partials. However, inharmonicity, as the phenomenon

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[134] Shirlaw, The Theory of Harmony. The 19th partial is mentioned by Ouseley according to Shirlaw, as the basis for the true minor third.

[135] Perrett, op. cit., p. 149.

[136] See Young's article in Grove's Dictionary of Music and Musicians, Tuning, Piano.

is called, affects octave replicas of lower partials as well, a fact which has not destroyed the role of the octave in music, or even of the quadruple octave. Inharmonicity is cumulative as the distance from the fundamental increases, and therefore is relatively slight in its effect on the superparticular ratios (that is, ratios comprised of numbers only one removed from each other such as 13:12). If 12:11 is deemed an aesthetically desirable phenomenon, the fact of inharmonicity need not affect its achievement. Psychologically, however, the knowledge of inharmonicity is likely to prove a handicap to favorable consideration of the upper partials; one can already hear the question "Why consider them when they are bound to be out-of-tune anyhow?"

Despite the unquestionably satisfying melodic character of the series 8:9:10:11:12, the addition of the 11th partial to musical systems involves such complications that venturesome pioneers such as Partch who really use it will probably long continue to be outnumbered by those who cite it to justify such diverse phenomena as 12- (Schönberg), 19- (Yasser), and 24- (Wyschnegradsky) tone temperaments whose substance, in most cases, has little to do with the 11th partial. Nevertheless, so broad are the possibilities of multiple division that, as Partch has shown by example, it affords ample room to those who would explore partials as high as 11, and possibly even higher.

THE ACOUSTICAL POSITION OF 12-TONE TEMPERAMENT

Every interval in 12-tone temperament has, as the foregoing shows, been attacked at one time or another. The octave has been attacked for being too perfect; the fifth both for being too perfect and not perfect enough. The major third has been attacked in theory for being too large, but in practice it has often been made still larger. Finally, 12-tone temperament has been attacked for making no provision for intervals involving the numbers 7 and 11.

12-tone temperament was more a discovery than an invention, since its properties were inherent in the condition of "12-ness" long before Werckmeister proposed the "well-tempered" scale. Werckmeister may, in fact, not be the decisive advocate of 12-tone equal temperament at all[137] but, whatever the case, its qualities reflect a fundamental approach of his, which is to regard as most important among the ratios those which are closest to unity.[138] Whatever can justly be said against 12-tone temperament it represents with greatest fidelity those intervals whose ratios

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[137] Dupont and most of the traditional sources cite Werckmeister as the principal exponent of 12-tone temperament, but Barbour holds that Werckmeister never recommended precisely equal temperament at all, but rather a rough approximation. The question is not essential here, because, in the totality of the field of possible tuning systems, Werckmeister's and 12-tone equal temperament cannot be far apart. And 12-tone temperament does realize Werckmeister's premise.

[138] "Je näher eine Zahl der Unität je vollkommener dieselbe ist." Quoted by Dupont.

most closely approach unity. Example 21 shows how the discrepancies in the representation by 12-tone temperament of small-number intervals increase as the highest prime number in the ratio increases.

Example 21: Discrepancies in 12-tone Temperament

Interval class:	Average discrepancy in cents	Average discrepancy in % of possible error
Prime class 2: 2:1	o	o%
Prime class 3: 3:2 (inversions not shown)	2	4%
Prime class 5: 5:4, 6:5, 9:5	14	28%
Prime class 7: 7:4, 7:5, 7:6, 9:7	30	60%
Prime class 11: 11:8, 11:7, 11:6, 11:9, 11:10	40	80%

The closeness of 3:2 to 7/12 is quite remarkable, especially when one considers that in order to find an equal temperament with an approximation of the fifth which is closer, one must proceed as high as 29 tones to the octave. Therefore, if one holds, with J. Murray Barbour and Fritz A. Kuttner that[139] "superparticular ratios beyond the limit 7:8 cannot have musical value . . . there is even reason to doubt the musical merit of a superparticular ratio as low as 5:4," then 12-tone temperament seems more than an adequate system but rather almost a perfect one. It is to those who regard a close approximation to the 5th

[139]Notes to The Theory of Classical Greek Music, op. cit., column 10.

partial or any kind of approximation of the 7th partial as desirable that 12-tone temperament must appear inadequate. For such musicians as desire closer approximations for the 5th and 7th partials or higher, multiple division of the octave is justified on acoustical grounds.

THE ACOUSTICAL POSITION OF 19-TONE TEMPERAMENT

A shift from 12- to 19-tone temperament would involve the sacrifice of some of the purity of the fifth in return for thirds which are much closer to just thirds than their counterparts in 12-tone temperament. The intervals of the 7th partial are represented more faithfully in 19- than in 12-tone temperament when measured in cents, but only very slightly so when measured on the possibly more realistic basis of percentage of the maximum possible error. Example 22 shows the discrepancies of the groups of basic intervals in 19-tone temperament and compares them to the corresponding errors in 12-tone temperament.

There is improvement over 12-tone temperament for all interval classes except the first, not only in the total size of the discrepancy but also in the percentage of possible error. The improvement is insufficient to provide a satisfactory basis for the 7th or 11th partials. It is in the region of the 5th partial that 19-tone temperament scores its most important gains. At the expense of the loss

Example 22: Discrepancies in 19-Tone Temperament

Interval class	Average discrepancy in cents	Improvement	Discrepancy in % of maximum possible error	Improvement
Prime class 3: 3:2	7	-5	24%	-20%
Prime class 5: 5:4, 6:5, 9:5	5	9	14%	14%
Prime class 7: 7:6, 7:5, 7:4, 9:7	15	15	48%	12%
Prime class 11: 11:6, 11:7, 11:8, 11:9, 11:10	23	17	74%	6%

of the great accuracy of its fifths, 19-tone temperament eliminates the rather large errors in the 12-tone thirds and can easily, therefore, be considered an acoustical improvement . . . or setback depending on one's point of view.[140]

[140] For a more detailed treatment of this subject see Chapter 12.