CHAPTER 13

GENERAL DISCUSSION OF METHODS FOR CHOOSING A SYSTEM OF MULTIPLE DIVISION AND SOME CONCLUSIONS

Elliott Carter, in speaking of the problem of diversity within contemporary music,[1] makes a point which is completely valid when applied to multiple division. Carter states that the composer "is free . . . to do what he likes . . . but . . . whatever he chooses to do, radical or conservative, his music will further divide into small sub-groups the handful of people who will listen to contemporary music at all. . . . While diversity of opinion is much to be welcomed, where so little support exists such decimation of interest, one hesitates regretfully to conclude, can lead to cancelling of efforts and ultimately to their negation."

In what field could this be more true than multiple division, where, in spite of the very limited public attention and support which is accorded, the few active composers and theorists work in seemingly contradictory directions? With such limited support is there room for just tunings and equal temperaments, for systems using and not using the upper partials, for systems using quarter-tones and systems not using quarter-tones, for positive systems and negative systems, for systems using a few more

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[1]*Musical Quarterly*, April 1960, pp. 200-1.

than twelve tones and systems using many more than twelve tones, for systems employing modifications of current instruments and systems demanding new instruments? In such a broad field, is there a system or group of related systems whose superior merits could gain general approval if they were generally known? Would such a system or group of related systems have any chance of becoming generally known? What kinds of research projects or attempts at composition or instrument construction or modification might help to bring about that state of affairs wherein musicians of all kinds might know what they need to know in determining the next general system of intonation, if indeed there is to be a next system of intonation? It is the purpose of this concluding chapter to deal (mostly indirectly) with questions of this kind. The method to be employed will be to try to associate specific acoustical and aesthetic premises with the systems of multiple division which best meet these premises and to attempt to identify accurate processes through which such associations can be made. I shall not try to persuade anyone to a specific point of view since it is my conviction that there is not yet enough experimental evidence to warrant an attempt to close ranks behind any one system of multiple division.

There are four fundamental questions of aesthetic values which, it seems to me, must be answered by each individual interested in multiple division and concerned with finding that system which will most appeal to him. With his answer to each question he can alter the substance of each of the succeeding questions.

Question One: Is there any acoustical validity to small-number ratios as the basis for a musical system? If so, which small-number ratios are to be included unconditionally among those with acoustical validity for a musical system, which ones conditionally, and under what conditions? This is probably the most fundamental question facing music today, and it is probably farther from having a generally accepted answer today than at any previous time in the history of the art. This question was taken up interval by interval in the first chapter; a summary of the present views of major composers and theorists is as follows: The Cologne school and other groups representing similar views apparently hold that no small-number ratio possesses any special intrinsic musical value. The rest of the musical world seems to remain convinced of the value of at least one small-number ratio, the octave, although several proposals have been made to modify its size slightly.

The serial dodecaphonists reject all small-number ratios except the octave. As serial dodecaphony possesses a substantial following among composers and musicians, their views on small-number ratios can not be rejected outright. However, since serial dodecaphonists, of all groups of knowledgeable musicians, tend to be the least inclined toward the altering of the present tuning system,[2] their views can be dismissed for the purpose of the study of multiple division as not germane to the subject. Except for the serial dodecaphonists, all musicians appear to consider 3:2 and 4:3 to have fundamental importance. The fifths of many non-Western musical cultures are considerably distorted, sufficiently so to cause debate as to whether the fifth is or is not an element of all musical systems. Let it suffice that it is an essential consonant element in all Western musical systems, past and present, except those cited above.

At this point anything approaching unanimity ceases completely. The remaining three questions will be framed in such a way as to assume an affirmative answer to question one as it involves ratios of the numbers 1 through 4, since the advocates of fixed systems of multiple division appear to be completely in accord on this point (Yasser is the only possible exception).

Question two: Is it preferable to have equal temperaments, or is it preferable to use exact small-number ratios (besides the octave)? If question one, above, is the most fundamental question dividing musicians today, question two is the most fundamental question dividing advocates of

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[2]See, for example, the quotation from Babbitt, above, at the beginning of Chapter 3.

multiple division of the octave (who tend to find themselves in agreement on a fundamental level regarding question one). If this study has tended to stress the work of the advocates of equal temperament it is because it is easier to analyze and explain their systems and to relate them to one another. Until now, with the exception of several of the Pythagorean systems, systems of multiple division not involving a basis in equal temperament have been strictly individual affairs, leading to no formation of schools and to no accumulation of works by more than one composer. The emphasis in this study should not be construed as indicating any negative judgment upon the works of the advocates of untempered systems. Perrett, Partch, Groven, and their predecessors have made important contributions to musical theory and practice.

Question two can also be put as follows: are approximations valid as representatives of small-number ratios? If this can be answered affirmatively a second question can also be asked: are the advantages of equal temperament sufficient enough to warrant the substitution of approximations of pure intervals for the real thing? The formation of all melodic and harmonic combinations possible within a system of any and all tones of the system is, of course, possible only in an equal temperament. Music, tonal and atonal alike, has relied on this resource for so long that the retraction of such a possibility might seem to many an intolerable retrogression. In any case, as Partch himself recognizes, the discontinuation of the principle of equal temperament would represent a far more revolutionary change in the fabric of music than the change from one temperament to another.[3] The untempered systems based on Pythagorean tunings must be excepted from this generalization, because as mono-intervalic systems they resemble equal tempered systems, and because their basic interval, the fifth, is so closely approximated in 12-tone temperament.

Among the untempered systems of multiple division which have been used, Perrett's 19-, Partch's 43-, and the Pythagoreanists' 24- and 36-tone systems have been noted in particular. To this might be added a 41-tone untempered system which Louis Greensway apparently employed in at least one work performed during World War II.[4] It might be noted that every one of the numbers cited above has been proposed by others as the basis for an equal temperament. It is rather curious that there is a numerical association between systems based on such completely differing concepts of proper intonation. A glance at the chart in which Partch's and Sauveur's 43-tone systems are compared (Example 39) shows how great the differences between them are.

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[3]Partch, _op_. _cit_., pp. 305-6.

[4]The only data I have on this is an announcement of a coming concert of 41-tone untempered music by Greensway in Philadelphia, March 13, with year not given.

Perrett's component intervals vary from 27 to 85 cents, and his system can hardly be said to resemble 19-tone temperament.

Nevertheless, it may not be entirely a coincidence that their systems employ the same number of tones per octave as important equal-tempered systems. Both Partch and Perrett "fill out" their systems to correct the largest variations in size among the component intervals. It is by this process that each reaches his final number of tones. Partch adds 14 to what was originally a 29-tone system; Perrett adds 5 to what was initially a 14-tone system. In a sense these men are, by this process, attempting "from the other side" to find the same equilibrium between perfection and approximation which the advocates of equal temperament are seeking. While the advocates of equal temperament insist on perfect equalization within the system and seek to approximate pure intervals as closely as possible, their counterparts insist on the perfection of the pure intervals and seek to approximate equality between the intervals. But these two processes tend naturally to lead to the same number of tones, because departure and error are complementary aspects of the same phenomenon. If with a given number of tones an equal temperament can be constructed with relatively small error in the approximation of the pure intervals deemed important, these same pure intervals can be used in an untempered system with equally small departures from the intervals of the equivalent tempered system. It should finally be noted that the number of tones present in Partch's and Perrett's systems before the additions of tones needed to equalize the intervals, represent far less desirable numbers for equal-tempered systems than the numbers of tones employed by the final systems.

To some extent, then, the problems involved in choosing the number of tones for an equal temperament apply also to choosing the best number of tones for an untempered system. Beyond this, very little can be said about the latter, as the possibilities are almost infinite. This is even evident in the vast number of "just" systems shown in the previous chapter as bases for 19-tone temperament. It would seem far more difficult to expect large numbers of theorists and composers ever to close ranks behind a single specific just system than behind a single equal tempered system which might continue to mean different things to different users. It is necessary, owing to the foregoing, to concentrate on the equal-tempered systems in attempting to find answers to the remaining questions, even though these answers may apply indirectly to untempered systems as well.

Question Three: What degree of approximation will be accepted as tolerable for each of the intervals present in a tuning system, and what methods can be used to evaluate degrees of "satisfactoriness" in the approximation of various intervals within a tempered system? With untempered systems a similar question must be posed concerning the degree of difference in the size of the constructing intervals that will be tolerated.

It is with this question that the crux of the problem of choice within the field of multiple division, narrowed down to equal temperament, is first met. Answers have varied from a refusal to accept even an error of two cents in the perfect fifth to Barbour and Kuttner's statement that commatic errors are no longer disturbing.[5] If a musician is able to name which small-number ratios he deems essential to his system and the maximal error which he will tolerate in his system, he can be told the number of tones in the simplest equal-tempered system that will meet his demands. If however, he wishes to use a graduated series of tolerances, whereby some intervals must be more closely in tune than others, the problem becomes more difficult.

Another complicating factor is the role of divergence in the sizes of the errors. A single system may contain particularly good approximations of some intervals to compensate for barely tolerable approximations of others. Theorists have debated over the proper method for evaluating

[5]Introductory notes for The Theory and Practice of Just Intonation, col. 31.

(1) Squares

the relative error in different systems involving more than one approximated interval. A method favored by many is calculation by the sum of the squares of the discrepancies. The discrepancies are calculated in logarithmic units such as cents and the squares of each taken. The system having the lowest sum of the squares of the intervals of discrepancy is then called the best system. Fokker cites this method to support his advocacy of 31- and 53-tone temperaments. According to Fokker, M. P. K. Ligtenberg, whose figures he uses, used the squares of the intervals of discrepancy (calculated in units of 1/100 octave or 12 cents), rather than the intervals themselves, simply to nullify the mutually cancelling effect of positive and negative signs.[6] This could, however, be done quite arbitrarily simply by disregarding the signs. The use of squares of the logarithmic values of the intervals can produce results drastically different from the results obtained by the use of the logarithms of the interval ratios themselves. The use of squares tends to favor the systems involving relatively equal discrepancies over those whose average discrepancy may be smaller but which contain one somewhat larger discrepancy.

The use of squares leads to a judgment in which the largest error among the basic intervals of a system is

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[6]. . . since both positive and negative numbers possess positive squares. Fokker, La Gamme..., p. 156.

taken into greater account than the other errors, but not into exclusive account. This seems a logical and reasonable solution to the dilemma caused by intervals involving errors of different sizes. But it leaves unsolved problems arising from differences in the importance of the intervals considered. What if the largest discrepancy involves the least important of the intervals considered? The interval having the largest error may indeed be justifiably considered of special importance because of that error, but there may be other factors tending to emphasize the importance of other intervals.

It is possible to emphasize the perfect fifth, as many theorists have done. Possibly because of the conditioning effects of the long use of 12-tone temperament, many surveys have shown musicians to be far more sensitive to slight deviations in the intonation of the perfect fifth than to similar deviations in thirds or other intervals not as well represented in 12-tone temperament. With the more distant partials, one encounters conflicting theories as to whether it is more or less important that they be represented as accurately as the lower partials. Partch insists, for example, that the 7th and 11th partials create consonant effects sufficiently subtle to be spoiled by even small errors in intonation, while other theorists claim to see the seventh partial (and even the 11th partial) as present in the 12-tone tempered system.

The sum of squares method, as it has been applied to date, has shown no allowances for inequalities in the importance of the accuracy of the various intervals involved. Woolhouse, in 1835, used the method with three intervals, the perfect fifth, the major third, and the minor third. Fokker applied the same method to the perfect fifth, the major third, and the natural seventh. Wedell and Bertelson applied the same method, although Kornerup does not tell us what intervals they used.[7] Elsewhere they applied the sums of simple deficiencies without squares to 35 intervals within each system, involving one for each letter name with natural, flat, sharp, double-flat, and double-sharp.[8]

Below are the results of Ligtenberg's and Wedell-Bertelson's computations, as reported by Fokker and Kornerup, respectively.

Ligtenberg-Fokker
(in 1/100 octaves squared)

10-tone temperament	7.4663
12-tone temperament	8.1533
15-tone temperament	4.1802
17-tone temperament	3.8901
22-tone temperament	1.6447
31-tone temperament	0.2044
41-tone temperament	0.2682
46-tone temperament	0.3318
53-tone temperament	0.1646
56-tone temperament	0.3304

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[7]Kornerup, Acoustic Valuation of Intervals, p. 7. His citation is of articles by Wedell and Bertelson in Musik, Copenhagen, 1917 and 1919.

[8]Kornerup, Acoustic Methods of Work, p. 27.

Wedell & Bertelsen-Kornerup
(in cents, not squared)

12-tone temperament	1174 cents
19-tone temperament	458 cents
31-tone temperament	174 cents
50-tone temperament	67 cents
Pythagorean (12-tone)	1780 cents.

It is quite evident that Ligtenberg and Wedell are looking for two quite different things. Except for 12- and 31-, the temperaments deemed worthy of examination and listing are completely different. Agreement is reached only on (1) the excellence of 31-tone temperament which is held in high esteem by Wedell and Bertelson, Ligtenberg, and Kornerup and Fokker alike; and on (2) the particular weakness of 12-tone temperament, which Ligtenberg shows to have an even greater deficiency than 10-tone temperament, owing to the relatively good 7th partial in the latter system. Were Ligtenberg to evaluate 19-tone temperament by his method, the deficiency would be approximately 3.7 square centioctaves.

Question Four: Assuming that a means can be found to evaluate the various deficiencies of a temperament, what relative importance is to be given to the difficulty entailed by the number of tones employed? Musicians have offered answers radically at variance with one another. This element, which might be called the difficulty factor, is hardest of all to evaluate. Is the difficulty of 12-tone temperament close to the limit of the ability of human hands to play and human ears to absorb? Does doubling the number of tones in a system double its difficulty, or does it come nearer to quadrupling the difficulty? Or, on the contrary, would it require a quadrupling of the number of tones to effect a doubling of the difficulty? These questions are very difficult even to attempt to answer without accumulated experimental data, and their documented answers would represent a major contribution to the study of the feasibility of many systems of multiple division. It is only when reliable information on this subject is available that it will be possible to make a reasonably valid decision on "how high to go" in the number of tones per octave to insure the maximum possible accuracy of intervals without overtaxing human artistic capacities.

In order to simplify this problem for the present, it will be useful to divide the field of multiple division into distinct areas of complexity. It is of little value to compare the acoustical merits of 19- and 53-tone temperaments, for example, because the difficulty factor is so different in the two cases. The boundaries between the areas of complexity must be arbitrary, of course, but I have tried to select them as logically as possible. Since we are used to thinking in a 12-tone pattern, I have selected twice 12, or 24, as the upper limit of the first area of complexity. Besides being a multiple of 12, 24 commends itself as an upper limit because above it there is a void of advocated systems, none being mentioned until 29- and none being widely advocated until 31-tone temperament. The first and second areas of complexity are therefore clearly separated.

For an upper limit to the second area of complexity I have selected 36 for precisely the same reasons. 31-, 34-, and 36-tone temperaments are the principal systems in this group. Because of the special credentials of 53-tone temperament, I have extended the limits of the third area of complexity to include this system. While admitting the possible injustice to such systems as 41- and 43-tone temperaments by the extension of this category to 53, I feel justified as it represents less than a 50 percent increase over 36 tones whereas every other category represents an increase of 50 percent or more.

All systems involving more than 53 tones to the octave will, for practical purposes, be discounted by being considered as lying within a fourth area of complexity.

Within each of the areas the demands upon a system of temperament are properly quite different. Within the first area one might reasonably expect some acoustical improvement over 12-tone temperament in addition to the obvious increase in harmonic diversity rendered possible. Within the second area a considerable increase in the accuracy of represented intervals and in the number of partials involved might reasonably be expected. Within the third area excellent accuracy and/or multiple partials would seem an altogether reasonable expectation.

METHODS OF CHOICE: TRIAL AND ERROR

Owing to the finite limits on the number of equal temperaments within the first groups, and to the extent of the explorations of these temperaments which have already been completed, it is quite possible to select one's desired temperament through elimination by trial and error. Following this procedure, it is probably best to begin by eliminating all equal temperaments in which the perfect fifth is not obtained with a sufficient degree of accuracy, then to repeat the process for each of the other essential intervals in the order of their importance. Owing to the different demands on the temperaments in each area of complexity, they are considered separately.

Area 1: The errors for the fifths in each of the equal temperaments in this area are shown in Example 52 in cents and in percentage of possible error.

To find the best means of evaluating the relative differences of the two kinds of error shown below again requires a value judgment on the role of the difficulty factor. Accepting a compromise view in order to have some basis for evaluation, I shall evaluate the intervals according to the simple product of the two factors which

Example 52: The Fifths in Area One

Number of tones	Size of fifth	Error-cents	Percent of possible error
12	700	2.0	04
13	738.5	36.5	79
14	685.7	16.3	38
15	720	18.0	45
16	675	27.0	72
17	705.9	3.9	11
18	733.3	31.3	94
19	694.7	7.3	22
20	720	18.0	60
21	685.7	16.3	57
22	709.1	7.1	26
23	678.3	23.7	91
24	700	2.0	08

I shall call the Combined Error Factor. The seven best temperaments in area one from the standpoint of their fifths only are:

System	Combined Error Factor
12	.08
24	.16
17	.43
19	1.61
22	1.85
14	6.19
15	8.10

One can draw the line where one chooses on the tolerability of the fifth. Probably the two most logical places are either somewhere between 17- and 19-tone temperaments for a strict limit, or between 22- and 14-tone temperaments for a loose limit. 14- and 15-tone temperaments seem out of the question if the approximation of the perfect fifth is to be of any importance whatsoever. So are all other temperaments in area one.

The thirds for the five temperaments shown above to have discussable fifths are:

Number of tones	Size of major thirds	Error-cents	Error-percent
12	400	13.7	27
24	400	13.7	55
17	352.9	33.4	93
19	378.9	7.4	22
22	381.8	4.5	16

The consideration of the major thirds tends to invert the relative values shown for the fifths. 17-tone temperament would have to be eliminated from any further consideration if thirds are to be held of any importance whatsoever. The Combined Error Factor for the major thirds of the remaining four temperaments in area one is shown below.

Number of Tones	Combined Error Factor for Major Thirds
22	0.62
19	1.63
12	3.70
24	7.53

If we now consider the combined error factor of the perfect fifths and the major thirds, the result varies according to the process we employ. We can use the product of the errors to favor the single very small error. This is a worthless method, however, since it is the larger rather than the smaller error that poses the greater danger to a tempered system. We can use the sum of the errors, or the sum of the squares of the errors. The latter process emphasizes the importance of the larger error.

We can weigh the two intervals unequally if we so desire. In the following chart, the combined error factors of the two intervals are placed in various combinations.

Example 53: Combined Errors of Fifths and Thirds

Method Used	12-	19-	22-	24
Product	0.3	2.6	1.2	1.2
Sum	3.8	3.2	2.5	7.7
Sum of squares (approximate)	14.0	5.3	3.8	56.0
Sum with fifth rated 2 to 1 over third	3.9	4.8	4.4	7.9
Sum of squares with fifth rated 2 to 1 over third	14.0	7.9	7.2	56.0
Sum with fifth rated 3 to 1 over third	4.0	6.5	6.2	8.0
Sum of squares with fifth rated 3 to 1 over third	14.0	10.5	10.6	56.0

According to the method used, 22-tone temperament is shown to be the best in some of the measurements, 12- in others, and 19- in yet another. 24-tone temperament runs last in all but one measurement.

The extremely favorable showing of 22-tone temperament in the statistics compiled above is dissipated considerably if the minor third, 6:5 is also taken into account. The errors of the minor third in each of the remaining temperaments are as follows:

Number of tones	Size of minor third	Error-cents	Percent
12	200	15.6	31
19	315.8	0.2	1
22	327.3	11.7	41
24	300	15.6	62

The combined error factor for this interval is as follows:

12	4.84
19	0.00
22	4.80
24	9.68

The addition of the minor third, even with very slight weight, turns the results strongly in favor of 19-tone temperament. For example, considering a weighting of 4-2-1 involving the sums of the squares of the errors of fifth, major third and minor third respectively, the result proves to be:

12	19	22	24
51.3	15.8	37.5	202.0

The same weighting, using the simple sums of the combined error factor multiples produces the following values:

12	19	22	24
12.6	9.7	13.4	25.3

Consideration of the harmonic seventh in the first area of complexity is probably out of the question, as is demonstrated by the errors of the interval 7:4 in each of the four surviving temperaments in this group.

Number of tones	Size of seventh	Error-cents	Percent of possible error
12	1000	31.2	62
19	947.4	21.4	68
22	981.8	13.0	48
24	950.0	18.8	75

The prospects of 22-tone temperament are thereby improved somewhat with respect to those of the other temperaments, but an error as great as 13 cents in a temperament involving intervals as minute as those of 22-tone temperament is a highly questionable one.

Conclusions on area one: There is no equal-tempered system with 24 or fewer tones with satisfactory coexistant intervals for the 3rd, 5th, and 7th partials. If the 3rd partial alone is to be admitted, 12-tone temperament retains its hegemony. If the 5th partial is to be admitted only in the form of the major third, 22-tone temperament appears the best, with 19-tone temperament as close second. If, however, the minor third is also to be considered, 19-tone temperament is substantially the best of the temperaments in this area.

<u>Area 2</u>: Within area two, the only temperaments whose fifths are sufficiently accurate to merit further consideration are 29-, 31-, 34-, and 36-tone temperaments. 29- and 31- are prime numbers and their systems are therefore entirely new. 34-tone temperament contains the same good fifths as 17-tone temperament, along with a great improvement in its thirds. 36-tone temperament contains no improvement on the thirds of 12-tone temperament despite the great increase in complication involved. Below is a comparison of the fifths in the four systems in this group which contain possibly adequate fifths.

Number of tones	Size of fifth	Error-cents	Percent	Combined Error Factor
29	703.5	1.5	7	.11
36	700	2.0	12	.24
34	705.9	3.9	22	.85
31	696.8	5.2	26	1.35

With the perfect fifth taken alone as the basis of a temperament, 29-tone temperament replaces the multiples of 12 as best. But it is highly doubtful that an increase in complication into the second area can be justified on the basis of the perfect fifth alone. Following are the figures for the major third.

Number of tones	Size of major third	Error-cents	Percent	Combined Error Factor
29	372.4	13.9	66	9.5
36	400	13.7	82	11.2
34	388.2	1.9	11	0.2
31	387.1	0.8	4	0.0

The error for the major thirds of 29- and 36-tone temperaments is so great that once that interval is considered, even if weighted lightly as compared with the fifth, 31- and 34-tone temperaments emerge as the only satisfactory prospects in this area. Of the two, 34-tone temperament appears to be the better on the basis of its fifths. When the minor third is brought into consideration, the superiority of 34- to 31-tone temperament becomes more pronounced.

Number of tones	Size of minor third	Error-cents	Percent	Combined Error Factor
31	309.7	5.9	30	1.8
34	317.6	2.0	11	.2

Within area of complexity two, among systems built on fifths and thirds only, 34-tone temperament appears a clear winner, involving no error larger than 4 cents. It would appear under the circumstances to be a very much underrated system for, although it is mentioned favorably among other temperaments by many writers, it appears to have obtained the specific advocacy of none. This may be due in part to the non-cyclic character of the fifths. Bosanquet's rule for forming thirds from fifths does not apply; the third of 34-tone temperament is never obtainable by a series of fifths from the tonic. 34-tone temperament possesses the further liability of an unequal partition of the major third into whole tones whose sizes relate to one another as 6 to 5. 6 + 5 = 11

With the consideration of the harmonic seventh, the balance swings heavily to 31-tone temperament. Below are the figures for the interval 7:4.

Number of tones	Size of seventh	Error-cents	Percent	Combined Error Factor
31	967.7	1.1	6	0.1
34	952.9	15.9	90	14.3

It is especially because of this last set of figures that 31-tone temperament thrives today in Holland and in many theoretical works, while 34-tone temperament is ignored. The former system possesses the 7th partial in very close approximation, the latter does not possess it at all. Once the 7th partial is brought under consideration, all competition for 31-tone temperament in this area of complexity vanishes.

With the addition of the 9th partial, and especially of the 11th, 31-tone temperament proves distinctly inadequate, leaving no adequate system in the second difficulty area.

Area 3: The relative merits of the temperaments in area three can also be shown by trial and error. The process can be simplified for the reader, however, because of the towering dominance of 53-tone temperament at every stage of consideration involving fifth (error less than 0.1 cent), fifth and third (the major third is about 1.4 cents too small), and fifth, third, and seventh (although the seventh is nearly 4.8 cents too large and taken alone does not make an especially good showing in this temperament). It is only when the 11th partial is added to the less-than-perfect 7th that the system appears to have passed its reasonable limits (error of the 11th partial, 8 cents or about 70 percent).

41-tone temperament is of some note in this group because it is the first temperament to approximate all of the first 11 partials within six cents. The thirds are the weakest intervals in this temperament; the minor third is a trifle over six cents too large, while the major third is nearly six cents too small. The fifths are excellent, and the 7th partial is less than three cents deficient. With interval distinctions as minute as those of 41-tone temperament, a 6-cent error must be considered quite substantial, but as this is the only temperament in the third group admitting the possibility of all partials through 11, it is worth some consideration. In both 41- and 53-tone temperaments the diatonic scale can be produced either with equal whole-tones and Pythagorean major thirds or with unequal whole-tones and thirds which approximate just thirds.

Conclusion on Tempered Systems as Obtained by Trial and Error: The 7th partial being unattainable in the first area of complexity with the lower partials also represented, the most complete system would represent all intervals in the senario as closely as possible. This is 19-tone temperament. In the second area, 31-tone temperament dominates by its inclusion of the 7th partial together with a reasonable approximation of the intervals in the senario. In the third area, 53-tone temperament is excellent in its rendition of the senario but weak on the upper partials. 41-tone temperament is the first instance where the 11th partial might play a role in an equal temperament together with the lower partials. A rather poor 5th partial mars this otherwise excellent temperament.

MATHEMATICAL METHODS

Even while using trial and error in their selection of temperaments theorists have attempted to find a more scientific basis for choosing one temperament over another. Since trial and error has provided all of the information needed by most musicians in the selection of a system of equal temperament from approximations of the pure small number ratios, one might be justified in asking why mathematical methods need be employed at all. Besides the obvious advantage a workable mathematical formula would possess in saving calculating time for a theorist who does not have available all of the figures he might need for comparison and proof, there are, it seems to me, other advantages to using a mathematical formula. An accurate and significant mathematical formula speaks with an authority which no system supported only by trial and error can achieve. It may well be the lack of this very authority which has prevented multiple division from occupying a prominent place in that part of the modern musical world which is preoccupied in certain respects with mathematics. The search for mathematical authority behind systems of multiple division has met, in most cases, however, with very little success. The formulas have tended to derive their existence from favored systems rather than the reverse. And the formulas when applied in series often have led to patently inferior temperaments. Nevertheless, at least one hitherto unpublicized method has led to a large measure of success, and a workable union between mathematical formulae and musical systems may well at long last exist.

Mathematical methods for the determination of the most desirable temperament have been varied; many different mathematical formulae have been explored, even by single authors. Barbour surveys several of these in <u>Tuning and</u>

Temperament, beginning on page 128. I shall discuss a few which have been proposed by the current disciples of multiple division, concentrating on those not mentioned by Barbour.

Yasser and Kornerup arrive, by different methods, at what amounts to a mathematical formula for the determination of an equal temperament. The desirable temperaments must all, according to their theory, belong to the Fibonacci series 2-5-7-12-19-31-50.... To Yasser the reasons are more sociological than mathematical, while Kornerup relates his choice of a Fibonacci series closely to his Golden Tone theory. It is true that the golden ratio, ω , equals the limit, as $n \to \infty$ of $\frac{n+1}{n}$ where n and n + 1 are consecutive members of a Fibonacci series.

The existence of 5- and 7-tone temperaments among Eastern musical systems, the current use of 12-tone temperament, and the favorable prognosis of 19- and 31-tone temperaments place the theory of a Fibonacci series as the mathematical basis for desirable temperament in a most favorable light. From the standpoint of evolution, the Fibonacci theory is also attractive, because it enables a temperament to develop, as it were, out of the two immediately preceding systems in combination. It is this aspect of the Fibonacci theory which is so attractive to Yasser.

Nevertheless, it is very hard to see how the Fibonacci series theory can be held to have any mathematical validity. The golden mean is an admirable act of artifice, but it does not derive its sanction from the harmonic series which appears more irregular in its favors.[9] However much Kornerup may insist that the ideal fifth is 696 cents, the ideal fifth in the real world of acoustics is 702 cents. That 31-tone temperament contains exactly the combined number of tones of 12- and 19-tone temperaments does indicate that it will possess intervals representing a mean of those intervals of the two simpler systems. But this does not guarantee that it will be a better system than the two simpler ones. This latter event occurs only if the errors of the two simpler systems tend to cancel one another out. Since the major thirds of 19-tone temperament are too small, while the thirds of 12-tone temperament are too large, one might expect that the thirds of 31-tone temperament will represent an improvement over both, and they do. The success of the members of the Fibonacci series up to 31 is the result of the opposition of positive and negative intervals in the 5- and 7- systems. This opposition creates a pendular balancing relation from system to system, up to 31-tone temperament, which tends to make each successive system more accurate in its representation of the principal acoustic values than the one preceding. At this point, however, the characteristics of the limit

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[9]For a detailed discussion of this subject, see Chapter 9, above.

values for both fifth and third prove too small, especially in the case of the fifth. Neither in its fifth, its third, nor its seventh, is the next member of the series, 50-tone temperament, an improvement over 31-tone temperament. A comparative chart of the errors of these three values is shown below:

	31-tone		50-tone	
	size	error	size	error
3:2	696.8	5.2	696.0	6.0
5:4	387.1	0.8	384.0	2.3
7:4	967.7	1.1	960.0	8.8

Further demonstration of the failure of the Fibonacci series on the 50-level lies in the comparison of the errors of 50- and 53-tone temperaments. Through the seventh partial, the latter interval is better for every possible interval except for 7:6, where the excesses of 50-tone temperament's 3rd and 7th partials tend to cancel one another out.

The supporters of adducing temperaments by the Fibonacci series are right in their basic premise that the best temperaments with large numbers of tones are based on the adding together of the numbers of tones of already good temperaments. But their theory neglects to insure that the systems so combined will possess the opposite characteristics which cancel one another out in the additive process. 19-

tone temperament, a negative temperament, complements far better 34-tone temperament which is positive, than 31-tone temperament which is also negative. Hence 19 plus 34 produces 53, an excellent temperament, while 19 plus 31 produces 50-, a decidedly inferior temperament from the standpoint of the small number ratios.

Würschmidt's use of mathematics in formulating his theory of rational tone systems is described above in Chapter 1.

Fokker makes many uses of mathematical formulae as they relate to musical systems. His article, Les Mathematiques et la Musique, explores many procedures for measuring a temperament, including a mode of evaluation in which unity is the sum of the squares of octave, fifth, third and natural seventh. His chief contribution to the art of selecting a temperament by mathematics is in his use of analytical geometry to ascertain the number of tones in a temperament from the properties of the defining intervals.

The various defining intervals are arranged according to the number of fifths, thirds, and sevenths, so that they will equal either zero or an exact octave or multiple of an octave. The syntonic comma, for example, which Fokker uses as a defining interval in a number of instances, is expressed as follows: four fifths minus one third (equals two octaves), or 4, -1, 0 as a function of its fifths, thirds, and sevenths, respectively.

Fokker proceeds to derive that temperament which uses the following three defining intervals: the syntonic comma; 225:224 (two fifths plus two thirds, minus a seventh); and 1029:1024 (three sevenths plus a fifth). A system based on these three defining intervals is expressed in analytical geometry as follows:

Interval	Fifths	Thirds	Sevenths
81:80	4	-1	0
225:224	2	2	-1
1029:1024	1	0	3

In analytical geometry the solution is found by adding all products of groups of three numbers that would be formed by the descending diagonals if the formula were continued, and deducting from it all similar products of ascending diagonals. Thus, the equation above becomes (24 + 1 + 0) - (-6 + 0 + 0) equals 25 + 6 or 31.

Of course, with other data the same mathematical process suggests other temperaments. Substituting the schisma for the syntonic comma, the pattern becomes

8	1	0
2	2	-1
1	0	3

which is equal to 41. Using the first two intervals above, as in 41-tone temperament, but substituting 6144:6125 (the difference between three thirds plus two sevenths, and the perfect fifth) for 1029:1024, 53-tone temperament is obtained by the formula

$$\begin{matrix} 8 & 1 & 0 \\ 2 & 2 & -1 \\ -1 & 3 & 2 \end{matrix}$$

The system Fokker has demonstrated cannot be used until the defining intervals have been established. The establishment of defining intervals is an intermediate process between the establishment of acoustical standards and their realization in a temperament. Attempts to eliminate this intermediate procedure have, however, been made.

Barbour has made such an attempt with ternary continued fractions. By virtue of what Barbour calls a "mixed expansion" he is able to obtain the best of the systems using fifths and thirds as a series.[10] Barbour's performance is as ingenious as it is complex. He first applies the standard or Jacobi ternary continued fractions, obtaining a series of supposedly ideal temperaments using thirds and fifths. Finding the series badly wanting, he reverses the terms, obtaining another expansion on the same prin-

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[10] Barbour, Music and Ternary Continued Fractions, American Mathematical Monthly, 1948, p. 545.

principle: better but still not entirely satisfactory.[11] Finally he works out a formula whereby the two processes are alternated, the choice between them to be determined by which of the two would involve the larger devisor. The result of this complicated procedure is almost identical with Brun's somewhat similar calculation based on ternary continued fractions but using subtraction instead of Barbour's division. Barbour's method is available to all who are interested; as Brun's is generally unknown in this country, it will be examined in considerable detail.

PROFESSOR BRUN'S METHOD

The most impressive mathematical method yet proposed for the derivation of systems of equal temperament from a series of just intervals (or from any other intervals if they should be desired) was first presented in 1919 by Viggo Brun, a Norwegian mathematics professor. He has since expanded on his views in a series of articles, the most recent and complete of which will appear this year in the Nordisk Matematisk Tidskrift. I am convinced that it provides the music theorist with the most useful apparatus

[11] The Jacobi expansion tends to favor temperaments with excellent thirds but questionable fifths: 3, 25, 28, 31, 87. The last two terms listed have good fifths as well. The expansion through the reversal of terms favors the fifths: 1, 2, 5, 12, 41, 53... The mixed expansion alternates on a principle similar to Brun's where the largest of the possible terms is used for each new stage of the expansion.

yet devised for probing the reaches of multiple systems of equal temperament.

Brun's method employs Euclidean algorithms. To begin with, the logarithms of all just intervals on which the system is to be based are computed. Any number of intervals may be employed (although Brun concedes that the method begins to lose its accuracy if too many factors are involved) and any members of the harmonic series may, for experimental purposes be omitted. Let us follow Brun's process, using the octave, fifth, and major third as values for our example.

He calls the logarithms of the three values a, b, and c respectively, and places them, in order from the highest to the lowest, in the left hand column. The right hand column shows values for x_1, y_1, z_1, x_2, y_2, z_2, and x_3, y_3, and z_3, as shown in Figure 1. These represent the number of times the ratio whose logarithm is on the left will go exactly into the ratios of the intervals x, y, and z, which in this case represent the octave, fifth, and major third. Figure 2 shows the same formula with numerical values.

FIG. 1

	x	y	z
a	x_1	y_1	z_1
b	x_2	y_2	z_2
c	x_3	y_3	z_3

FIG. 2

(2:1)0.3010	1	0	0
(3:2)0.1761	0	1	0
(5:4)0.0969	0	0	1

Brun's basic process is to substitute "a - b" for "a" in the left hand column, producing with each stage logarithms representing smaller intervals, while in the right hand columns he substitutes "x_1 plus x_2" for "x_2," which is to say that he increases the number of times the intervals (now smaller) are used. Throughout the process, a, b, and c represent the sizes of intervals while x_1, x_2, and x_3, etc., represent the number of occurances of each of the respective intervals such that ax_1 plus bx_2 plus cx_3 equals exactly an octave, ay_1 plus by_2 plus cy_3 a perfect fifth, etc.

The substitution shown in Figure 3 represents no basic change in the values, since $(a - b)x_1$ plus $b(x_1 + x_2) = ax_1 - bx_1 + bx_1 + bx_2 = ax_1 + bx_2$, exactly the same as in Figure 1.

FIG. 3

a - b	x_1	y_1	z_1
b	x_1+x_2	y_1+y_2	z_1+z_2
c	x_3	y_3	z_3

FIG. 4

0.1249	1	0	0
0.1761	1	1	0
0.0969	0	0	1

The smallest interval, c, is a kind of carried-over remainder which remains outside of the main calculations until one of the other figures becomes smaller, whereupon it is renamed b, and the new smallest term becomes the remainder. In Figure 4, the numbers in Figure 2 are modified according to the process shown in Figure 3. In place of the octave, log (a - b) representing the perfect fourth, appears. The x column to the right of the line shows that one fourth plus one fifth comprises an octave. In Figure 5, the terms of Figure 4 are simply rearranged to place the three intervals in positions a, b, and c according to size. In Figure 6, the basic process is repeated. The fourth, b, is deducted from the fifth, a, leaving "a - b," a 9:8 major second. The octave is shown to consist of two fourths and a major second, the fifth of one fourth and a major second. The third is yet to play a role in the process.

FIG. 5

(3:2)0.1761	1	1	0
(4:3)0.1249	1	0	0
(5:4)0.0969	0	0	1

FIG. 6

0.0512	1	1	0
0.1249	2	1	0
0.0969	0	0	1

In Figure 7, the terms in Figure 6 are rearranged to place the interval sizes in descending order. In this process, the major third is advanced to position b. Figure 8 shows a repetition of the basic procedure, with the diatonic semitone 16:15 resulting from the subtraction of 5:4 from 4:3. As this is the smallest interval so far produced, it assumes position c in the rearrangement which is depicted by Figure 9.

FIG. 7

(4:3)0.1249	2	1	0
(5:4)0.0969	0	0	1
(9:8)0.0512	1	1	0

FIG. 8

0.0280	2	1	0
0.0969	2	1	1
0.0512	1	1	0

In Figure 9, the octave is shown as comprised of two just major thirds, a major tone, and two diatonic semitones. This can be stated as follows: $\frac{5}{4} \times \frac{5}{4} \times \frac{9}{8} \times \frac{16}{15} \times \frac{16}{15} = \frac{2}{1}$. In Figure 10, it is further broken down into traditional units of the just diatonic scale, three 9:8 tones, two 10:9 tones, and two 16:15 semitones. The other right hand columns show the traditional breakdowns of third and fifth among these intervals.

Figure 11 shows the necessary rearrangement to restore the values on the left hand side to numerical order, and Figure 12 shows the next step in the subdivision with a new small value, the syntonic comma, emerging to assume the role of "c" for quite some time.

FIG. 9

(5:4)	0.0969	2	1	1
(9:8)	0.0512	1	1	0
(16:15)	0.0280	2	1	0

FIG. 10

(10:9)	0.0457	2	1	1
(9:8)	0.0512	3	2	1
(16:15)	0.0280	2	1	0

FIG. 11

0.0512	3	2	1
0.0457	2	1	1
0.0280	2	1	0

FIG. 12

(81:80)	0.0055	3	2	1
(10:9)	0.0457	5	3	2
(16:15)	0.0280	2	1	0

We are now at the point where recognizable tempered systems are about to emerge, and it is time to ask ourselves how one is to interpret the figures obtained. As has been stressed, $ax_1 + bx_2 + cx_3$ will always equal an octave. Do we, then, add the number of terms in the first column to the right of the line to obtain the number of tones in our temperament? No; this is not done, because the third value is external to the main process and is likely to be so small as to be negligible to the process as a whole.

In Figure 13 (which is a simple rearrangement of Figure 12) the octave is composed of ten units, three of which are mere syntonic commas. To propose a 10-tone temperament with units representing everything from minor tones (10:9) to syntonic commas (81:80) would be absurd. What Figure 13 does offer, in its top row, (derived from row 2 of Figure 12) is an instance of 5-tone temperament in which the fifth and major third have their proper values. A comparison of this row with the sum of the terms in Figure 9, shows the superiority of this process. The sum of the terms in Figure 9 are 5, 3, and 1 respectively. If this were indeed supposed to represent 5-tone temperament, the third would be much too small. 386 cents is considerably closer to 480 cents than it is to 240 cents; the row 5, 3, 2 representsthis temperament better than the rougher, earlier estimate, 5, 3, 1.

FIG. 13

(10:9)	0.0457	5	3	2
(16:15)	0.0280	3	1	0
(81:80)	0.0055	3	2	1

Figure 14 carries Brun's process a step farther, with a new temperament of 7 tones to the octave emerging.

Figure 15 represents the characteristic rearrangement of terms, and Figure 16 carries the process another step, showing the emergence of 12-tone temperament. It might be mentioned at this point that the progression from 5- through 12- has begun to follow Yasser's Fibonacci series. This it will continue to do as long as every new interval value (a - b) remains larger than the syntonic comma, which is presently value c. As soon, however, as the new value has been reached which is smaller than the syntonic comma, the syntonic comma will pass from the role of "defining" interval to that of "constructing" interval and the values to the right of it in the chart will play a role in determining the number of tones in the temperament. It is Figure 15 which best shows the 12-tone system under Brun's approximation. In the octave are seven diatonic semitones (16:15) and five chromatic semitones (25:24). Three syntonic commas are "left over," which are generally added to three of the diatonic semitones in the theory of just intonation.

FIG. 14

0.0177	5	3	2
0.0280	7	4	2
0.0055	3	2	1

FIG. 15

(16:15)0.0280	7	4	2
(25:24)0.0177	5	3	2
(81:80)0.0055	3	2	1

FIG. 16

0.0103	7	4	2
0.0177	12	7	4
0.0055	3	2	1

Figure 17 represents the rearrangement of Figure 16 in the descending order of the intervals, and Figure 18 represents the next step: 19-tone temperament. The syntonic comma remains the smallest figure, but the new interval, representing the difference between the chromatic semitone and the diesis of the major thirds, 128:125, is nearly as small.

Figure 19 shows the rearrangement, and Figure 20 the next step, whereby 31-tone temperament is shown, and where a new interval, smaller than the syntonic comma is formed. It is this stage which alters the Fibonacci series, as 81:80 assumes position b in the rearrangement which is shown in Figure 21.

Thus it is not 19, but rather 3 tones, representing the number of syntonic commas carried by the simpler systems as a remainder, which must be added as the syntonic comma is advanced to the rank of constructing interval. Figure 22 shows the emergence

FIG. 17

.0177	12	7	4
.0103	7	4	2
.0055	3	2	1

FIG. 18

.0074	12	7	4
.0103	19	11	6
.0055	3	2	1

FIG. 19

.0103	19	11	6
.0074	12	7	4
.0055	3	2	1

FIG. 20

.0029	19	11	6
.0074	31	18	10
.0055	3	2	1

FIG. 21

.0074	31	18	10
.0055	3	2	1
.0029	19	11	6

of 34 as the next system through the combining of 31- and 3-tone temperaments, the latter shown, by Brun's method, to have the best features to balance the weaknesses of 31-tone temperament.

Figure 23 shows the rearrangement which follows; a new smallest interval assumes position c. In Figure 24, it is 53-tone temperament which is shown to arise out of the combination of ultrapositive 34- and negative 19-tone temperaments. With the evolving of 53-tone temperament, the syntonic comma has completed its role as constructing interval and will itself be subdivided.

Figure 25 represents a rearrangement and Figure 26 introduces the next temperament of the series, possessing 87 tones to the octave. The excellent merits of this system are considered briefly in Chapter 6, above.

FIG. 22

.0019	31	18	10
.0055	34	20	11
.0029	19	11	6

FIG. 23

.0055	34	20	11
.0029	19	11	6
.0019	31	18	10

FIG. 24

.0026	34	20	11
.0029	53	31	17
.0019	31	18	10

FIG. 25

.0029	53	31	17
.0026	34	20	11
.0019	31	18	10

FIG. 26

.0003	53	31	17
.0026	87	51	28
.0019	31	18	10

The next temperament to emerge, after the rearrangement shown in Figure 27, is 118-tone temperament in Figure 28. This is the "perfect system" of so many advocates of temperaments based on fifths and thirds, and it is also shown in Chapter 6.

FIG. 27

.0026	87	51	28
.0019	31	18	10
.0003	53	31	17

FIG. 28

.0007	87	51	28
.0019	118	69	38
.0003	53	31	17

The algorithm can be continued, in which case it yields 205, 323, 441, and 494-tone temperaments. The temperaments agree almost completely with those derived by Ariel and by Barbour by different methods, with the latter members of the series showing the greatest divergence between them. The error of third and fifth alike in 441-tone temperament is less than .05 cents and compares well with the errors in systems of many tones which are advocated by other theorists.

APPLYING BRUN'S PROCESS TO OTHER INTERVALS

A process similar to the one outlined above can be followed for any combination of intervals. The theorist may choose to use additional values in the harmonic series. Brun himself has carried the process through using the above values plus the seventh partial, obtaining as temperaments for the octave, 13-, 15-, 18-, 31-, 35-, 53-, and 68-tones.[12] The agreement of this series with the one studied above at the 31- and 53- levels strengthens the case for these temperaments. Brun has also used the same process with all partials through 11, and obtained the following results: 15, 17, 22, 37, 41, 63, 72.... Of these, his own preference lies with 41- and 72-tone temperaments, both of which have been found quite attractive by other theorists, usually for very different reasons.[13] Carrying this algorithm farther, I find the next temperaments to be 89-, 161-, 233-, and 270-tone temperaments. The intervals of 270-tone temperament show the potential value of Brun's method for locating exceptional temperaments.

Interval	3:2	5:4	7:4	11:4
Size of interval	702.22	386.67	968.89	551.11
Size of error	0.27	0.36	0.[illegible]	0.2

The smallest unit of the system being 4.44 cents, the error never exceeds 18 percent of the possible; this is a very good achievement for a system with so many disparate factors.

It is also possible to use Brun's method with

[12] In my own calculations using Brun's methods, I arrive at slightly different figures: 49 instead of 35, and 71 instead of 68.

[13] 41-tone temperament is discussed above in the first section of Chapter Five, and 72-tone temperament is discussed near the end of Chapter Four.

intervals lying outside of the harmonic series, should it be the conviction of anyone seeking a more complex equal temperament that substitute intervals (such as a larger major third) are preferable. The method can also be used with members of the harmonic series but omitting one or more, as Yasser suggests with regard to the 3rd partial. Brun's procedure, in short, provides a means for converting any set of preferred intervals into a series of equal temperaments tending to increase in accuracy as more tones are added. There is a shortcoming, in that there is no room for differentiating the importance of the various intervals involved. This same weakness has been noted in other methods for calculating advantageous temperaments. A further shortcoming, if one wishes to call it that, is that it still does not perform a musician's thinking for him. He must still decide which intervals he is to use, and he must select from the series which the method he employs yields that single system which is the best compromise, for his purposes, between accuracy and expediency.

CONCLUSION

If one accepts the acoustical validity of small-number ratios on the one hand and the principle of equal temperament on the other, one appears all but bound to conclude that the most desirable temperaments belong to the following series: 12, 19, 22, 31, 34, 41, 53, 72.... The underlined tempered systems seem, in particular, to offer distinct acoustical refinements over all those which precede them in the series. The systems not underlined display important disadvantages as well as some advantages over those which immediately precede them. In general, accuracy and scope increase with number of tones, but so does difficulty. It is possible to measure the degree of accuracy of a temperament minutely, and we have attempted to do so in the foregoing pages. It is only possible to guess at the difficulties involved in learning to perform and to hear music based on 19-, 31-, 53-, or 72 sounds to the octave. Certainly the generalized keyboard deserves a generalized test.[14] Ear-training experiments with selected subjects may afford a partial answer to another important aspect of the question, "how difficult?"

The generalized keyboard might afford a beginning for a new tempered system; the ear might be expected gradually to digest the new pitch relationships manually produced. With the mastery by the ear would come the availability of the vocal and string media as well as the slide trombones. Valved brass instruments could be easily adapted by the addition of one or two microtonal crooks. Keyed wind instruments would present the greatest problem,

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[14] See Appendix 1, below, for diagrams and descriptions of proposed generalized keyboards.

but multiple tiers of keys on a wind instrument are hardly out of the question, and their exploitation has only just begun with the present woodwind instruments.

There seems little point in adapting many instruments until the direction of further evolution, if there is one, becomes clearer. The absence of sufficient information for an imminent decision must be conceded. There appears, therefore, to be little chance that multiple division will replace 12-tone temperament in the near future.

But there would appear to be no reason why multiple division on an experimental scale shouldn't co-exist with 12-tone temperament. There is ample justification for further experimental composition, experimental hearing tests, and experimental attempts at obtaining performance virtuosity in various systems, not only in the isolated studies of a Partch, a Fokker, and a Wyschnegradsky, but wherever creative musicians are dissatisfied with the materials currently at their disposal.