CHAPTER 5

EQUAL TEMPERAMENTS BASED ON FIFTHS AND THIRDS

Section 1: Positive Systems

This chapter introduces a domain of rather complex theories and counter-theories. Bosanquet's General Theory of the Octave,[1] offers a valuable procedure for relating and measuring the various systems encountered in this phase of the study. His theory possesses both the advantages and defects inherent in the use of such familiar landmarks as our 12-tone tempered system and the diatonic scale as a basis for comparison.

Bosanquet's contribution to the field of multiple division of the octave is almost too immense to evaluate. His writings have influenced as diverse a group as Fokker, Kornerup, and Partch, and his own efforts at instrument-construction provided a major breakthrough and an excellent example for those who followed him.[2] His General Theory of the Octave supplies a precise language for dealing with

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[1] An Elementary Treatise on Musical Intervals and Temperament, 1876, p. 60.

[2] The instrument for which Bosanquet is justly famous is a harmonium normally tuned to 53-tone equal temperament, and possessing a "generalized keyboard" whereby for any acoustic grouping or relationship of sounds, the spatial relationship between the keys to be depressed remains exactly constant, regardless of transposition.

multiple divisions based on fifths and thirds. His interests covered the whole broad range, acoustical, historical, and anthropological, of subjects related to the multiple division of the octave.

BOSANQUET'S GENERAL THEORY OF THE OCTAVE

Bosanquet's theory begins with several useful definitions which will be cited and then observed throughout this and subsequent chapters.

> Regular Systems are such that all their notes can be arranged in a continuous series of equal fifths.[3]
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> Regular Cyclical Systems are not only regular, but return to the same pitch after a certain number of fifths. Every such system divides the octave into a certain number of equal intervals.
>
> Error is deviation from a perfect concord.[4]
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> Departure is deviation from an equal-tempered interval. Unless the equal-temperament is otherwise specified, departure is measured from the intervals of 12-tone equal temperament.
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> Intervals taken upwards are called positive, taken downwards, negative.
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> Fifths are called positive if they have

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[3]Nothing is said, under regular systems, about the necessity of the fifths meeting. Both Pythagorean intonation and meantone tuning are regular systems by Bosanquet's definition. Equal-tempered systems are not only regular, but cyclical as well.

[4]Perfect concord meaning a concord based on a small-number ratio.

positive departures.[5]

Systems are said to be positive or negative according as their fifths are positive or negative.[6]

Regular cyclical systems are said to be of the r^{th} order, positive or negative, when twelve of the approximate fifths of the system exceed or fall short of the octave by r units of the system.[7]

The above list of definitions gives Bosanquet a yardstick, 12-tone equal temperament, which may influence his judgment, but which does not constitute a commitment to that effect. He now proceeds to formulate theorems pertaining to the nature of regular systems.[8]

Th. I. In any regular system, five 7-fifths semitones and seven 5-fifths semitones make up an exact octave.

By his interest in the two kinds of semitones, Bosanquet is concerning himself with the diatonic scale.

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[5]Note that departure rather than error is the basis for classification of the fifths. This is a most important aspect of Bosanquet's theory, as it establishes 12-tone temperament as the norm for comparison. The perfect fifth has, of course, a positive departure.

[6]This establishes 12-tone temperament as the mean between "positive" and "negative" systems.

[7]The concept of an order, r, may require explanatory illustration. Note that a unit of a system is its smallest interval. Let us take 19-tone temperament as an example. An approximate fifth in 19-tone temperament is 11 units of the system. Twelve such fifths total (12 x 11 =) 132 units. Seven octaves equal (19 x 7 =) 133 units. Therefore, in 19-tone temperament, "twelve of the approximate fifths of the system . . . fall short of the octave" by 1 unit of the system. In 19-tone temperament, $r = -1$.

[8]*Op. cit.*, p. 61.

Theorem I asserts that for any regular system the practice of determining two kinds of semitones from superimposed fifths is valid. This theorem is easily proved[9] by the identity 7 x 5 = 5 x 7.

> Th. II. In any regular system the difference between the 7-fifths semitone and the 5-fifths semitone is the departure of 12 fifths, having regard to sign.

This is self-evident, and is based on the simple arithmetical fact that 5 fifths plus 7 fifths equals 12 fifths and the assumption that the difference in the semitones will be created by the departure in the fifths since where there is no departure in the fifths (in 12-tone temperament) the two kinds of semitone are identical.

The next five theorems concern Bosanquet's theory of r, and are presented here without individual commentary. The concept of r further relates multiple division to 12-tone temperament, since for each value of r there is an endless series of regular cyclical systems, separated from one another by the addition or deletion of 12 tones to the

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[9] In 12-tone temperament, 12 semitones equals an octave. In any other system the semitones are altered according to a multiple of the departure of the fifth. As the size of the fifth is constant, the departures of seven 5-fifth semitones and of five 7-fifth semitones will each equal 35 times the departure of a single fifth. But because the 5-fifth semitone is decreased as the fifth is increased, while the 7-fifth semitone is increased with the fifth, the two 35-fifth departures will cancel out one-another, leaving only the 12 equal-tempered semitones which continue to make an octave.

octave.[10] The temperaments which have been considered in the preceding chapter would all be considered by Bosanquet to have r equal to 0. Temperaments of a given order of r can differ from one another quite extensively although they will possess some common characteristics as opposed to 12-tone temperament. Within a given order of r, the more tones there may be to the octave, the greater is the influence of 12-tone temperament on the intervals of the series (12-tone temperament is the limiting case of any order of r where n approaches ∞).

> Th. III. In a regular cyclical system of order plus or minus r, the difference between the 7-fifths semitone and the 5-fifths semitone is plus or minus r units of the system.
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> Th. IV. In any regular cyclical system, if the octave be divided into n equal intervals, and r be the order of the system, the departure of each fifth of the system is r/n semitones of 12-tone temperament.
>
> Th. V. If in a system of the r^{th} order the octave be divided into n equal intervals, r + 7n is a multiple of 12, and $\frac{r+7n}{12}$ is the number of units in the fifth of the system.
>
> Th. VI. If a system divide the octave into n equal intervals, the total departure of all the n fifths of the system equals r semitones of 12-tone temperament, where r is the order of the system.
>
> Th. VII. If n be the number of divisions in the octave in a system of the r^{th} order, then n + 7r will be divisible by 12, and $\frac{n+7r}{12}$ will be the number of units in the 7-fifths semitone.

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[10] When r = -1, for example, we have the endless series, 7-, 19-, 31-, 43-, 55-, 67-, 79-, etc.

Having explored the properties of the fifth, octave, and semitones, of the systems of the various orders, Bosanquet proceeds to theorems relating to other intervals.

> Th. VIII. Negative systems form their major thirds by four fifths up. Corollary: The departure of a third of a negative cyclical system n of order -r is $\frac{-4r}{n}$.

Barbour insists[11] that this formula is not valid where n is greater than 12. If Barbour is correct, all negative systems including such attractive and popular ones as 19- and 31-tone temperaments must be eliminated on the spot. Since the thirds of 12-tone temperament are produced in this manner and are substantially larger than 5:4, the negative systems which have been seriously proposed unfailingly improve the quality of the thirds.

> Th. IX. Positive systems form approximately perfect thirds by eight fifths down. Corollary: the departure of such third is $\frac{-8r}{n}$.

The application of this theorem, which makes use of the schismatic relationship, has a weakness. Since the third is produced by eight fifths, relatively slight deviations in the fifth are magnified 8 times in their effect on the thirds. 29-tone temperament affords a useful illustration, being a positive system of the order r = 1. In this tempera-

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[11] *Tuning and Temperament*, p. 130.

ment, the error of the fifth is only 1½ cents. The error of the third, however, consists of the schisma plus eight times the error of the fifth, a total of 14 cents, enough to render 29-tone temperament useless as a system with fifth and third.

> Th. X. Helmholtz' Theorem. The third thus formed with perfect fifths has an error nearly equal in amount to the error of the equal-tempered 5th.

This theorem involves only approximation and therefore cannot be proved. The two ratios which are semi-equated arise from different series, the one involving only 3 and 2, the other involving 5 as well, and their near-identity is fortuitious.

> Th. XI. In positive systems an approximate harmonic seventh can be obtained by fourteen fifths down.

> Th. XII. In negative systems an approximate harmonic seventh can be obtained by ten fifths up.

Owing to the number of fifths involved in each seventh, small variations in the fifths can effect huge displacements in the sevenths. Applied to 31- and 41-tone temperaments, theorems XII and XI, respectively, produce excellent results. 36-tone temperament, wherein $r = 0$, possesses an excellent harmonic seventh which cannot be obtained by any combinations of fifths at all. It is recalled from the previous chapter that the fifths in 36-tone temperament are not cyclic over the whole system, and the two members of the interval 7:4

belong to different cycles of fifths in 36-tone temperament which is, when all is said, an irregular system by the General Theory of Bosanquet.

WHERE R EQUALS 1

Bosanquet's classification of regular cyclical systems according to a value or *r*, positive or negative, affords a particularly useful perspective for examining the various possible temperaments. The minimal variation from 12-tone temperament is provided by temperaments in which *r* is equal to plus or minus 1. In view of the general concern with pure fifths and thirds in general, and the preservation of the acoustical advantages of 12-tone temperament in particular, it should come as no surprise that the vast majority of authors who have recommended multiple division have advocated systems in which *r* is no greater than plus or minus 1.

The order, *r* equals 1, includes the temperaments of 5-, 17-, 29-, 41-, 53-, and 65 tones per octave. In this order, the fifths are always larger than 700 cents, with 700 cents representing the limiting case as *n* approaches infinity. The operation of this principle is observed in the fifths of the six systems listed above, which are, respectively, 720 cents, 705.9 cents, 703.5 cents, 702.4 cents, 701.9 cents, and 701.5 cents. The great advantage of the systems of this order is their fifths, which are particularly excellent in the last four of the cited temperaments. The thirds, however, are often weak. Since the fifths are larger than in 12-tone temperament, the thirds formed by 4 fifths are even larger than the large thirds of present use, and are consequently impracticable. Following the procedure Bosanquet recommends (in Th. IX.), the thirds formed by 8 fifths vary considerably from system to system. In 5- and 17-tone temperaments the major third is, to all intents and purposes, a non-existent interval. In 29-tone temperament it is still 14 cents too small, an error prompting even Novarro, who experimented with the temperament at length, to discard it.[12]

In 41- and 53-tone temperaments, the third makes rapid improvement, being 6 cents too small in the former, and slightly more than 1 cent too small in the latter. When the temperament is increased to 65- per octave, the third becomes nearly two cents too large. Several theorists, among them Ariel and Würschmidt, but earlier Bosanquet,[13] have noted the opposite deviations of the thirds in 53- and 65-tone temperaments. Since both have excellent fifths, the temperament involving their sum of tones, 118-, must be nearly perfect from the standpoint of fifths and thirds.

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[12] Novarro, op. cit., pp. 162-4.

[13] Bosanquet mentions the performability of 118-tone temperament (not quite all of the tones) on his harmonium in his article, Temperament, or the Division of the Octave, from the Proceedings of the Musical Association of London, 1874-5.

All of the temperaments of the order r = 1, between 5- and 53-tones per octave, have been used or have been recommended. 5-tone temperament is the approximate tuning of the Indonesian salendro scale. 17-tone temperament has generated interest from time to time, and is often considered to be the properly constructed Arabian scale.[14] There have been at least two 20th century advocates of 17-tone temperament, Edmond Malherbe and Vladimir Karapetyan. Malherbe's views are published,[15] and he is seen to be interested in a 17-tone system for "third-tones" which he considers to have more validity here than in an 18- or 36-tone system. He may not even belong properly with the advocates of 17-tone temperament, for he asserts a leaning toward the Pythagorean intonation of his 17-tone system.[16] Perhaps his most interesting contribution is his suggestion for the notation of simultaneous sharps, flats, and/or naturals on the same line of the staff. This perplexing problem of contemporary music, aggravated by multiple division, he solves with the following, self-explanatory symbols.[17]

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[14] Barbour takes this view in Tuning and Temperament, p. 114.

[15] Le Ménéstrel, 19 July, 1929, p. 329.

[16] Ibid., p. 330.

[17] Ibid., p. 337.

Vladimir Karapetyan, a professor of engineering, argues for the use of 17-tone temperament on the grounds that it is a simpler expansion from 12-tone temperament than 19- and that it possesses better fifths than 19-tone temperament.[18] Barbour imagines that the larger-than-Pythagorean thirds and the small semitones might make 17-tone temperament excellent for melodic, though not for harmonic, purposes.[19]

29-tone temperament is the first of the multiple divisions with better fifths than 12-tone temperament. Novarro credits its first advocacy to Cheve. Novarro fretted a guitar to 29-tone temperament and conducted a series of experiments but discarded it, as had von Jankó and others, because of its weak thirds.

Von Jankó, famous primarily for his attempt to popularize a keyboard based on Bosanquet's multiple-division keyboard for general use in the 12-tone system, wrote one article on multiple division[20] in which 41-tone temperament is singled out for favorable attention. A number of major theorists, including Fokker, have given 41-tone temperament respectful attention as well. Like 17- and 29-tone tempera-

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[18]Letter from V. Karapetyan to J. Yasser, Aug. 22, 1935, unpubl.

[19]*Op. cit.*, p. 114.

[20]"Uber mehr als zwolfstufige gleichschwebende Temperaturen," *Beiträge zur Akustik und Musikwissenschaft*, 1901, p. 6.

ments, it contains a major third which is too small, but the error is, in 41-tone temperament, reduced to six cents. Besides superb fifths (less than half a cent too large), 41-tone temperament contains a natural seventh which is less than 3 cents too small, an excellent 9th partial, of course, and an 11th partial which is about as inexact as the major thirds. It is the first of the equal-tempered systems where the errors of all partials through 11 are sufficiently small to render the system discussable as a realization of all of these partials. Prof. Viggo Brun shows, by a process of algorithms,[21] how 41-tone temperament belongs properly to the mathematical series of temperaments which represent optimally the partials through 11.

Fokker considers 41-tone temperament quite seriously as a function of the partials 3, 5, and 7, without including 11. He goes to considerable effort to illustrate this system with a diagram of the three-dimensional tone-lattice as he conceives it.[22] He also charts the deviations of the various intervals based on small-number ratios in 41-tone temperament, measured in terms of units of the system. Example 26, an excerpt from Fokker's chart, shows the size of each of the intervals involving numbers 11 and smaller

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[21]Unpublished letter to the author, but note his publications in bibliography.

[22]Rekenkundige Bespiegeling der Muziek, section beginning p. 183. Similar charts from this book on 53- and 31-tone temperaments are reproduced, below.

and the errors of their approximations in 41-tone temperament. 1.000 equals 1/41.

Example 26: Small-Number Intervals in 41-Tone Temperament

	2	3	4	5	6	7	8	9	10	11
2	0	23.98								
3	.02	0	17.02	30.22						
4		-.02	0	13.20	-	33.10				
5		.22	-.20	0	10.78	19.90	27.80	34.77		
6				.22	0	9.12				35.85
7			-.10	.10	-.12	0	7.90	14.87	21.10	26.74
8				.20		.10	0	6.97		18.84
9				.23		.13	.03	0	6.23	11.87
10						-.10		-.23	0	5.64
11					.15	.26	.16	.13	.36	0

Fokker's final conclusion on 41-tone temperament is that it offers no acoustical improvement over 31-tone temperament to justify the complications involved in using 10 additional tones. He bases this conclusion on the figures of Ligtenberg, who used the sum of the squares of the deviations of fifth, third and seventh in arriving at his recommendations.

That 41-tone temperament offers no general improvement over 31-tone temperament is essentially true, since the 6-cent discrepancy of the thirds in 41-tone temperament replaces the 6-cent discrepancy of the fifths in 31-tone temperament. Nevertheless, to theorists such as von Jankó, who postulate a near-perfect fifth as the <u>sine qua non</u> for a temperament, 41-tone temperament does represent a distinct gain over 31-.

53-TONE TEMPERAMENT

With 53-tone temperament we arrive at what many theorists including Bosanquet have regarded as almost the utopian condition for music. Discovery of 53-tone temperament is generally credited to Mercator, the scientist famous for the map projection in which East is always East, and West is West, although Greenland becomes many times the size of Australia. ①

Yasser, however, cites the Chinese King Fang as the discoverer of the 53-tone system.[23] It is unlikely that either King Fang or Mercator thought of his discoveries in practical musical terms. They are both responsible for the discovery of the remarkable near-coincidence between 53 fifths and 31 octaves. The actual ratio for the interval separating the two is $3^{53}:2^{84}$ or:

$$\frac{19,383,245,667,680,019,896,796,723}{19,342,813,113,834,066,795,298,816} \quad ::$$

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[23] A Theory of Evolving Tonality, p. 31. Partch adds: "no royalty implied."

① This is a gross error. Nicholas Mercator and Gerhardus Mercator were two different people.

In recent years, Esther Tipple has attempted to revive interest in 53-tone temperament with a number of tables, charts, a slide-rule, and an instrument. The name of Mercator is again associated with the venture, as she has established the Mercator Foundation in Georgia, under whose auspices her work has been done.[24] Although she has suggested scales with various numbers of tones from within the system of 53-tone temperament, no known compositions have come from the Mercator foundation. Miss Tipple's 19-tone scale from within 53-tone temperament is noted below in chapter 12. p 331

Since the time of its mention by Mercator, 53-tone temperament has been considered by many theorists, usually favorably. Ariel includes it in his series of "good" systems, coming after 34-tone temperament. Würschmidt considers 53-tone temperament among the finest of the rational tone systems, devoting considerable space ot its analysis.[25] Ellis had earlier called 53-tone temperament among the three best[26] (together with 12- and 31-. Fokker was to select the same three temperaments). In 1912, the advocacy of 53-tone temperament appears in its most

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[24] Tipple, *Music Logarithmic Spiral* and other works, see bibliography. The address of the Mercator Foundation is given as Black's Bluff Road, Rome, Georgia.

[25] Würschmidt, *Die rationellen Tonsysteme*... *op. cit.*, p. 550 ff.

[26] Appendix to Helmholtz, *op. cit.*, p. 435.

suspenseful and mysterious form: an anonymous interview granted to G. Bender by a disciple of "Futurism" in Le Guide du Concert, who stresses that in 53-tone temperament Fxx (quadruple sharp) = Dbbbb. This is true.

It is to Bosanquet, however, that the greatest credit for the modern interest in 53-tone temperament belongs. He points out that not only is the fifth excellent in 53-tone temperament (its discrepancy is less than .1 cent) but the third is less than 2 cents too small, an error smaller than that of the fifth in 12-tone temperament. The single unit of 53-tone temperament is only slightly larger than the syntonic comma. Bosanquet points out that the syntonic comma equals 1/55.8 octave.[27] The discrepancy between 1/53 and 81:80 is slightly larger than one cent. The Pythagorean third is nearly 0.3 cents flat, and the error of the just major third is the sum of the two figures, 1.4 cents.

Bosanquet attempted to make practical reality out of 53-tone temperament with an instrument which may well prove to be the direct ancestor of any multiple division instrument in general use in the future.[28] That the 53-tone system is closely linked by Bosanquet to the various multiple just systems based on perfect fifths is evident

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[27] Elementary Treatise... op. cit., p. 13.

[28] Bosanquet's generalized keyboard is employed with slight modifications by Fokker for his 31-tone organ which has had considerable influence in recent years. A diagram of the generalized keyboards is included in the appendix of this study.

not only from his own text but also from the accompanying advertisement of the manufacturer of his instrument. The buyer who wishes an organ with multiple division but cannot afford the 53 tones or 84 keys demanded by Bosanquet may buy an organ with 24, 36, or 48 keys to the octave. The discrepancy between the tones of the systems with 24, 36, and 48 perfect fifths on the one hand and 53-tone temperament on the other is all but negligible. 53-tone temperament provides an approximation that will be in error by about 1 cent for every 12 fifths traversed in the Pythagorean systems. The discrepancy is larger, of course, if the fifths of the just system are tempered by 1/8 of a schisma to produce just thirds. It probably may be surmised that the organs in Mr. Jennings' advertisement were designed with perfect, rather than octa-schismatically altered fifths in mind.

The traditional diatonic scale can be played in two different ways in 53-tone temperament, as Steiner has pointed out.[29] With Pythagorean intonation, the whole-tones are produced by intervals of 9/53 and the semitones by intervals of 4/53. The discrepancy is negligible. A version of just intonation with an average deviation of less than a cent is obtainable with 9/53 for the major tone, 8/53 for the minor tone, and 5/53 for the diatonic semitone. For

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[29] Bericht von III Kongress, I.M.G., op. cit., p. 394.

musical requirements closely related to the diatonic scale, 53-tone temperament is little short of perfect. It may, however, be much too complex for so simple a use.

Among the theorists writing on 53-tone temperament, Wesley Woolhouse, in 1835, some 40 years before Bosanquet, complains about the inaccuracy of the thirds. Woolhouse, the initial proponent of 19-tone temperament, would hardly seem to be the man to be disturbed by an error of only 1½ cents. He does not state the size of the thirds or their error and it is possible that he calculated them incorrectly.

Partch, however, objects strongly, and with considerable justification, to the representations of the 7th and 11th partials in 53-tone temperament. The former is almost 5 cents too sharp, and the latter close to 7 cents too flat, the discrepancies being quite substantial in view of the fineness of the temperament.

The 5-cent error in the 7th partial notwithstanding, Fokker accepts this interval as an essential feature of 53-tone temperament, building a three-dimensional tone-lattice as is his custom with other multiple temperaments. His diagram is reproduced several pages hence in Example 28.

Neither Fokker, Öttingen nor Würschmidt, all of whom offer specific "just" intonations for 53 tones per octave, present any comparison between their proposed systems and 53-tone equal temperament. Example 27 shows the approximate tunings, in cents, of the 53-tone scales of these three as deduced from their diagrams, and compares them to Pythagorean and equal-tempered 53-tone systems. The intervals are measured from the middle-most tone in each diagram and the discrepancies likewise. The average deviation shown at the bottom involves all of the intervals as measured from a central tone, and is not the same thing as Barbour's mean deviation, which is concerned with the constructing intervals (units of the system).

Example 27: Comparison of 53-Tone Systems

	53-t temperament	Ottingen	Würschmidt	Fokker	Pythagorean
0	00.00	00.00	00.00	00.00	00.00
1	22.62	21.51		13.8	23.46
2	45.28	41.06		43.4	46.92
3	67.92	70.68		70.68	66.77
4	90.57	92.18		84.4	90.23
5	113.21	111.74		111.74	113.69
6	135.85	133.24		133.24	137.15
7	158.49	162.85		155.1	156.99
8	181.13	182.40		182.40	180.45
9	203.77	203.91		203.91	203.91
10	226.42	223.46		231.2	227.37
11	249.06	244.97		239.55	250.82
12	271.70	274.58		266.85	270.68
13	294.34	294.14		288.4	294.14
14	316.98	315.64		315.64	317.60
15	339.62	335.20	345.25	337.55	341.06
16	362.26	364.81	356.70	359.05	360.90
17	384.91	386.31		386.31	384.36
18	407.55	407.82		400.1	407.82
19	430.19	427.47		427.47	431.28
20	452.83	456.99		457.0	451.13
21	475.47	478.50		470.75	474.59
22	498.11	498.05		498.05	498.05
23	520.75	519.56		519.56	521.51
24	543.40	539.11		541.3	544.97
25	566.04	568.72		555.2	564.81
26	588.68	590.22		582.5	588.27

Example 27 (Continued)

	53-t temperament	Ottingen	Wärschmidt	Fokker	Pythagorean
27	611.32	609.78		617.5	611.73
28	633.96	631.28		631.28	635.19
29	656.60	660.89		653.2	655.03
30	679.25	680.44		674.65	678.49
31	701.89	701.95	701.95	601.95	701.95
32	724.53	721.50		729.25	725.41
33	747.17	743.01		737.6	748.87
34	769.81	772.53		777.53	768.72
35	792.45	792.18		786.4	792.18
36	815.09	813.69		813.69	815.64
37	837.74	835.19	843.30	(Six figures see below)	839.10
38	860.38	864.80	854.75	857.1	858.95
39	883.02	884.36		884.36	882.41
40	905.66	905.86		898.1	905.87
41	928.30	925.42		933.15	929.33
42	950.94	955.03		941.5	949.17
43	973.58	976.54		968.8	972.63
44	996.23	996.09		996.09	996.09
45	1018.87	1017.60		1017.60	1019.55
46	1041.51	1037.15		1039.5	1043.01
47	1064.15	1066.76	all others	1053.2	1064.86
48	1086.79	1088.26	same as	1088.26	1086.32
49	1109.43	1107.82	Ottingen.	1115.6	1109.78
50	1132.08	1129.32		1129.32	1133.24
51	1154.72	1158.94		1151.2	1153.08
52	1177.36	1178.49		1172.7	1176.54
53	1200.00	1200.00		1200.00	1200.00
Average deviation:		2.395	3.044	5.629	0.91
Average number of intervals used:		3.577	3.462	2.941	13.5

The six different pitches for Fokker's 37th tone are:

835.6
835.2
843.3
827.5
848.7
822.0

Example 27 reveals a number of characteristics of 53-tone temperament worth noting. As with 12-tone temperament it is the perfect fifth which is the dominating interval.[30] The very existence of more than one interpretation of just intonation shows that with 53 tones to the octave, as with 12, Barbour's statement is applicable: "There is no such thing as just intonation; but, rather, many different just intonations; . . ."[31] Barbour's statement continues, ". . . of these, the best is that which comes closest to the Pythagorean tuning." This is even more true of 53-tone temperament than of 12-, as the low figure under Average Deviation shows.

In general, 53-tone temperament would seem to be as good a system as most theorists have regarded it. The average deviations range from less than a cent, when compared to Pythagorean tuning, to around 2½ to 3 cents for the systems using fifths and thirds, to over 5½ cents for Fokker's system employing the natural seventh as well. By comparison the average deviations for various systems in 12-tone temperament are: Ptolemy's just intonation, 12 cents; Aron's meantone temperament, 13 cents; Pythagorean tuning,

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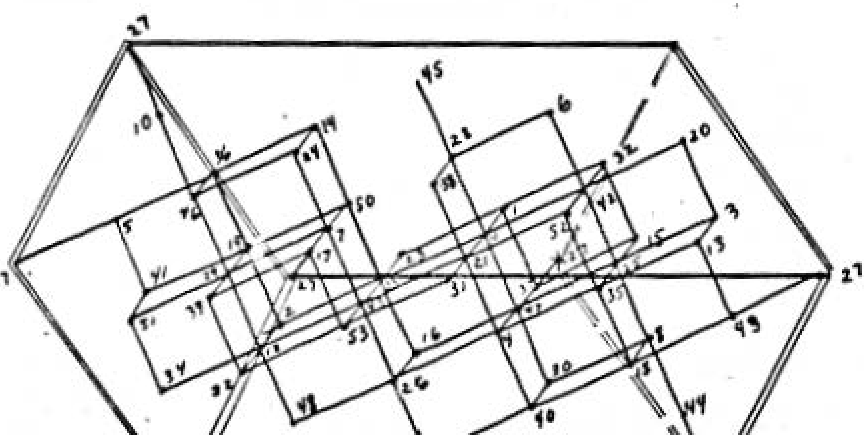
[30] Ariel, who likes to consider each of the "good" systems as related to the interval which is best represented by its temperament, acknowledges the hegemony of the fifth in 12- and 53-tone temperaments but not, of course, in 19- or 31-.

[31] *Op. cit.*, p. 105.

6 cents. The extent of improvement in 53-tone temperament is about proportional to the increase in the number of tones. The improvement is even better in the case of the Pythagorean system and in the case of systems involving septimal intervals. Septimally derived intervals in 12-tone temperament are all but unthinkable.

Of the four systems shown, Fokker's is the most interesting. Example 28 is a diagram of the system upon which the figures in Example 27 are based. I have transposed the numbers shown below by adding 10 to each of them, in order to make the tone shown by Fokker as 43 the generating tone of the system, since it is the most centrally located of the tones in his diagram.

Example 28: Fokker's 53-Tone System



In Example 27, there is shown for each system the average number of steps which must be taken to reach each member of the system from the central or generating tone. It can be presumed that some advantage would accrue to those systems which required fewer separate steps in relating one tone to another. It is here that Fokker's tri-dimensional system shows to its greatest advantage, for it involves an average of fewer than three separate steps to reach each tone from the generating tone even though it is not precisely at the center, whereas the bi-dimensional systems require more than 3. It is also here that the weakness of the uni-dimensional Pythagorean system is most apparent, with 13.5 steps separating the average member from the central tone. Würschmidt's system, slightly inferior to Öttingen's in intonation, is the more compact of the two, as can be seen by Example 27.

Fokker's system is suspended between six different representations of the 27th tone of his system (the 37th on our chart). It is symptomatic of the weakness of his system from the standpoint of intonation, that the six representatives of this one tone vary in pitch by 26.7 cents, more than the smallest unit of 53-tone temperament! The smallest interval between consecutive tones in Fokker's system is only slightly larger than 8 cents, while the largest is about 35 cents. It may well be justifiable to conclude that a tri-dimensional tone-lattice does not work well for a system in which one basic interval is very much better represented than another. The weaknesses of Fokker's system reflect the weaknesses of the 7th partial in 53-tone temperament.

In concluding our consideration of 53-tone temperament, it would seem reasonable to suggest that it is a superb temperament for fifths, a good temperament for fifths and thirds, and a questionable temperament once higher prime partials are brought into consideration. 53-tone being the most complicated equal temperament to be considered in any detail in these pages, this would appear the most suitable time to point out that no matter how minute the subdivision of the octave, just intonation always remains a step ahead of equal temperament. and they never can be brought together. 53-tone temperament also demonstrates that as long as there are two or more theorists considering a given musical system, there will always be contention over the just intonation.

WHERE R EQUALS 2

Four systems in which r = plus 2 are of sufficient value to merit some attention. Three of these have been proposed, 22-, 34-, and 118-tone temperaments, of which the latter will be considered in Chapter 6 in comparison with other temperaments involving a very great number of tones. In addition to the other temperaments already named, 46-tone temperament has acoustical features possibly worthy of consideration. Example 29 shows the size of the intervals most closely approximating 3:2, 5:4, and 7:4 in each of the three above-named systems, together with the discrepancies involved in each case.

Example 29: 22- 34- and 46-Tone Systems Compared

Interval	22-Tone	34-Tone	46-Tone
FIFTH	709.08	705.88	704.34
ERROR OF FIFTH	7.13	3.93	2.39
THIRD	381.8	388.24	391.3
ERROR OF THIRD	4.4	1.93	5.0
SEVENTH	981.8	952.9	965.2
ERROR OF SEVENTH	13.0	15.9	3.6

These systems compare not altogether unfavorably with those temperaments previously examined. The double order is reflected in the high degree of sharpness of the fifths, especially in 22-tone temperament.

Since the errors of the third and fifth in 22-tone temperament are in different directions, the error of the minor third is equal to their sum, almost 12 cents. This is probably excessive, but the system is otherwise within the realm of consideration, with its flaws about commensurate with those of 19-tone temperament. It is possible that the positive fifths, untried historically, might be more bothersome than the negative fifths of 19-tone temperament, which are also about 7 cents in error. The major third in 22-tone temperament is better than the major third in 19- (by a rather small margin), but the minor third is much worse. Neither temperament offers a close approximation to the natural 7th, although that of 22-tone temperament, erring on the same side of the seventh in 12-tone temperament but less than half as much, would sound relatively pure to our ears. 22-tone temperament possesses a familiar landmark, the tritone of 12-tone temperament. Opelt and Würschmidt report with some favor on this system, as does Novarro, who attributes its introduction to the East Indians. The 22 srutis of Indian music are not generally considered to represent an equal-tempered system, however.

34-tone temperament can be compared profitably with its negative equivalent, 31-tone temperament. The fifths are slightly better and the major thirds slightly less good. The minor thirds are better owing to the fact that both fifth and third are positive in error. Where 34-tone temperament is weakest in comparison to 31- is in its representation of the harmonic seventh. It is for this element that 31-tone temperament must be preferred to 34- for possible general use.

46-tone temperament improves the 7th partial of 34-tone temperament to a point where it might be useful, but the third is weakened considerably in the process. The great improvement of the fifths and thirds in 53-tone temperament (over 46-), achieved at only slight cost in the 7ths, renders the slightly more complex temperament the more likely to be useful.

Section 2: Negative Systems

Theorists have been reluctant to abandon the practice of building thirds by the superposition of four fifths. As the major third in 12-tone temperament is more than 13 cents too large, many theorists have built musical systems by reducing the size of the equal-tempered fifth.

WHERE R EQUALS -1

By far the most common order of these systems is that in which twelve fifths fall short of seven octaves by one unit. The limit of the fifth in this order, as in all orders based on 12-tone temperament, is 700 cents. This limit is approached from below, so that the fifth is always too small. This is a congenital weakness in the negative orders, one which has caused Drobisch to leave them out of his series of ideal temperaments[32] and Jankó to reject them out of hand. Even Bosanquet gives the negative systems a subordinate role.[33]

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[32] Drobisch, op. cit., p. 88.

[33] His harmonium is constructed primarily for positive systems, although it can be adapted for use with negative systems.

The series in which r = -1 begins with 7-, 19-, 31-, 43-, and 55-tone temperaments. All five of these systems have played a role in either musical practice or musical theory. The fifths in each of the five systems, in ascending order of n, are 685.7 cents, 694.7 cents, 696.8 cents, 697.7 cents, and 698.2 cents. The gradual improvement in fifths compensates in large measure for the added complications as n increases, but the improvement is too gradual to justify an unambiguous preference for those systems in this order where n is largest. The major thirds in the order approach a limit of 400 cents from below, crossing the desired 386.3 about where n = 31. The thirds for the five systems listed above are, in ascending order of n, 342.9, 378.9, 387.1, 390.7 and 392.7.

The harmonic seventh, as formed by ten ascending fifths according to Bosanquet's theorem (no. XI., see above), varies from 857.14 cents (more than 100 cents too small) in 7-tone temperament, to a limit of 1000 cents, very much too large. It is 21 cents too small in 19-tone temperament and 8 cents too large in 43-tone temperament. In 31-tone temperament, however, the harmonic seventh is approximated within 2 cents. It is the combination of excellent major thirds and natural sevenths which has made 31-tone temperament a particular favorite of many theorists.

The five principal temperaments of the order r = -1 will be taken up one by one, with 31-tone temperament reserved for the conclusion of this chapter, and 19-tone temperament reserved for a more detailed examination in subsequent chapters.

7-tone temperament is of significance more because of practice than theory. Siamese music is based on the approximation of this temperament. As will be noted subsequently, Joseph Yasser, one of the chief exponents of multiple division, sees in the Siamese musical system a direct ancestor of diatonic musical practice, frozen from further development by the arrival of equal temperament. Siamese musical practice is pentatonic, but the closed and complete system is essentially equal-tempered heptatonic.

If, as Fritz Kuttner has stated,[34] the fifths in Chinese music are consistently very flat, it is conceivable that some kinship may exist between Chinese musical practice and an approximation of 7-tone equal temperament. Such a kinship is merely speculative and is contradicted by other intonational practices in Chinese music; nevertheless, it would seem a better explanation for the phenomenon of the flat fifths than the dodecaphonic concept of the ld or the Pythagorean explanation usually given for the pentatonic scale.

7-tone temperament is of little value in Western music today, except for experimental purposes, since it would represent a retrogression from current practice both

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[34]Conversation, December 1960.

in complexity and in acoustical relationships. It is interesting to hear diatonic melodies played in 7-tone temperament, and very difficult not to conclude emphatically that the diatonic unequal scale is an aesthetic improvement. It is Yasser's contention that the abolition of 12-tone equal temperament in favor of a more complex system would provide more appropriate contours for 12-tone melodies much as 12-tone temperament provides a more suitable vehicle than 7-tone temperament for heptatonic melodies.

43-tone temperament is mentioned in most studies as the system proposed by Sauveur. It seems more likely, however, that Sauveur used equal temperaments only as a means of simplifying meantone systems for theoretical purposes and that he envisaged no music using 43 tones to the octave. Nevertheless, Sauveur's advocacy represents an endorsement of the acoustical attributes of 43-tone equal temperament since they are no different from those of the 1/5-comma temperament system Sauveur was actually recommending. It is in this temperament that the fifths and thirds yield equally to the demands of the other, rendering the temperament ideal for those who wish to preserve the relationship whereby the third is produced by four fifths without favoring either interval at the expense of the other.

Although cited in the standard works on multiple division, 43-tone temperament as a musical reality has yet to be advocated by anyone, unless Sauveur is considered to have advocated it. This may well be because of the compromises which characterize the system. The fine thirds and sevenths of 31-tone temperament are given up, as well as its relative simplicity, for what is in reality only a slight improvement in the fifths. This has hardly seemed worthwhile to most theorists in view of the almost perfect fifths in 53-tone temperament at the expense of only slight further complication. A 43-tone system without equal temperament has been advocated and used extensively by Harry Partch (see below, Chapter 7).

55-tone temperament is the complete system, part of which is represented in Silbermann's 1/6-comma temperament, much favored by many instrument makers and tuners around 1700 and, according to Sauveur, "in universal use by musicians"[35] at that time. According to Novarro, 55-tone temperament as such has had an advocate in Esteve. Its fifths represent a slight improvement over those in 43-tone temperament, provided at the expense of the thirds and of what might still remain of simplicity. It seems far more useful as the basis for a 12-tone meantone temperament than as a system to be used in its totality. As a meantone temperament it possesses the great advantage of thirds which are produced by four fifths. As a system used in its totality it possesses no such advantage, since how the thirds may be derived in an equal-tempered system is purely academic; the

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[35]Scherchen, The Nature of Music, p. 42.

only meaningful question concerns their quality. The chief intervals in 55-tone temperament compare most unfavorably with those in the positive system which is its most obvious rival, 53-tone temperament. Fifths, thirds, and sevenths are all distinctly purer in the latter, rendering the acoustical defense of 55-tone temperament an extremely difficult matter.

WHERE R EQUALS -2: 50-TONE TEMPERAMENT

Before proceeding to 31-tone temperament, I am considering 50-tone temperament, the only temperament of order r = -2 to have been seriously proposed, because its problems very closely resemble those of 55-tone and other first-order negative systems. Formed by the adding together of the numbers 19 and 31, the number 50 designates an equal temperament possessing the basic strengths and weaknesses of those two temperaments which are yet to be considered, with the significant added weakness created by the increased complexity of the system. Its fifths are almost 6 cents flat; this represents a deviation of over 48 percent of the possible error where the intervals are so small. The thirds are only 2.3 cents too small, but this is a greater error than that of the thirds in 53-tone temperament, a temperament whose principal virtue lies not in its thirds but in its fifths. 50-tone temperament represents the completion of Zarlino's

2/7-comma temperament and Robert Smith's 5/18-comma temperament. Its advocacy as a system of multiple division is generally attributed to Henfling, a contemporary of Sauveur, probably as erroneously as in the case of Sauveur and 43-tone temperament. It is probable that Henfling, like Sauveur, meant his system to be a basis for measurement (50 is a convenient number for that purpose) and the superstructure for a simple meantone system. Robert Smith advocated it in this way as later did Woolhouse.

Like 43- and 55-tone temperaments, 50-tone temperament seems most useful for the simple manner in which its thirds are derived from its fifths, a usefulness which is essential to the construction of a suitable meantone temperament using a small number of tones, but of no acoustical value in the complete cyclical system. It is 53-tone temperament, which forms its thirds and fifths independently of one another, and which cannot be used therefore as the basis of a twelve-tone meantone temperament, which otherwise completely dwarfs its near rivals in acoustical powers.

31-TONE EQUAL TEMPERAMENT

It is the negative systems which dominate the field of equal temperaments between 12 and 41. Two systems with order $r = -1$ lie within this area, 19- and 31-tone temperaments. These two systems have been advocated jointly by

Kornerup; Yasser has advocated the former for the present and the latter for the future. Würschmidt, Opelt and many of the other writers who have spoken with great favor of the one have done so equally of the other. Other writers, however, have made a more positive choice between the two systems. As will be demonstrated, most of the acoustical advantages lie with 31-tone temperament, while 19-tone temperament possesses those advantages which accrue from greater simplicity.

Holland and 31-tone temperament have maintained an affinity which stretches from the time of its discovery to the present. It was the Dutch mathematician, Christian Huygens, who demonstrated in 1732 that 31-tone temperament is a closed system representing the completion of Aron's meantone temperament. It was another Dutch scientist, Adriaan Fokker, retired professor and curator of the Teyler Institute in Haarlem, who built there a pipe organ tuned to 31-tone temperament which has become the central point in a recent surge of interest in that tuning system.

During the Renaissance, when Aron's ¼-comma temperament was in widespread use, a number of instruments were built using 31 tones to the octave. Best known of these was Vicentino's archicembalo, which appears to have had a companion instrument, an archiorgano. There is some confusion over Vicentino's exact intentions for the intonation of his instrument or instruments, a confusion which persists in the face of brave new attempts to pinpoint the pitches on the basis of Vicentino's own ambiguous instructions.[36] It is fairly well established, however, that at least the first orders of keys, and possibly the entire instrument, were tuned to ¼-comma temperament, rendering the arciorgano almost an equal-tempered instrument.

Several other 31-tone instruments were built shortly after Vicentino's. The patterns of flat and sharp tones on these instruments were complex and showed no regard for facility of modulation or transposition, a most understandable circumstance considering when they were built. One instrument, however, Fabio Colonna's "Sambuca Lincea," is said[37] to have had the first "generalized" keyboard, wherein each order resembled the one before it. According to Barbour, Colonna's orders are separated by 1/5 of a tone, and each order contains 7 tones to the octave. It would appear that Colonna's keyboard was only partly generalized, since 31 divided by 7 does not leave an interval that would be the basis for any musical system of the Renaissance or early Baroque.

31-tone temperament shares with other negative temperaments the quality of having excessively small fifths.

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[36]For the most recent and detailed examination of the tuning of Vicentino's instruments see Kaufmann, Henry W., Vicentino's Arciorgano; an Annotated Translation, Journal of Music Theory, April 1961, p. 32.

[37]By Barbour in Tuning and Temperament, p. 119.

The deficiency is nearly 5.2 cents, representing 27 percent of the largest error possible. Against this defect must be balanced a major third and a natural seventh of exceptional quality. The thirds are 0.8 cent sharp while the sevenths are 1.05 cents too small. Theorists have generally written favorably about this temperament,[38] but often with reservations about its fifths.[39] Some writers have found more cause for objection in the minor thirds (major sixths) than in the fifths.[40] The discrepancy for this interval in 31-tone temperament is 6 cents, the sum of the discrepancies of the fifth and the major third. Writers who have most emphatically endorsed 31-tone temperament while rejecting 19-, such as Fokker, have tended not to consider the minor third as an interval of the same rank of importance as the major third.

Woolhouse, on the other hand, is concerned with the minor third, as were Smith and Harrison before him. Since

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[38]Probably the most enthusiastic statement about 31-tone temperament was by B. S. Wedell, a Copenhagen actuary who occasionally wrote articles on music for Danish-language periodicals. Kornerup quotes Wedell as writing of 31-tone temperament that it is "an Eldorado for music theorists." Kornerup, Acoustic Valuation of Intervals, p. 7.

[39]Novarro, op. cit., p. 165-6, suggests 31-tone temperament for those types of instruments only, where beats, such as will be created by out-of-tune fifths, are not heard too pronouncedly.

[40]Robert Smith in the 18th century and Woolhouse in the 19th adopt this view. See Smith, op. cit., preface, and also page 121; and Woolhouse, op. cit., p. 49, where in spite of his reservations he says of 31-tone temperament: ". . . a very good scale . . . has been much approved of by many musicians."

the minor third is created by deducting the major third from the perfect fifth, the error in the minor third of any given equal-tempered system will be a function of the errors of the perfect fifth and major third. Where the error of one of these two determining intervals is positive and the other negative, as in 31-tone temperament, the error for the minor third will be equal to the sum of the errors for the perfect fifth and major third. Where, however, the errors of both of the determining intervals carry the same sign (as in 19-tone temperament, where both the perfect fifth and major third are negative), the error for the minor third will be equal to the difference between the errors of the major third and perfect fifth. In 19-tone temperament the errors for the major third and perfect fifth, although each quite substantial, cancel one another almost completely, leaving the minor third extremely close to perfection. This near-perfection is of significance to Woolhouse, but is of no importance to Fokker, since the minor third is to him not a generating interval.

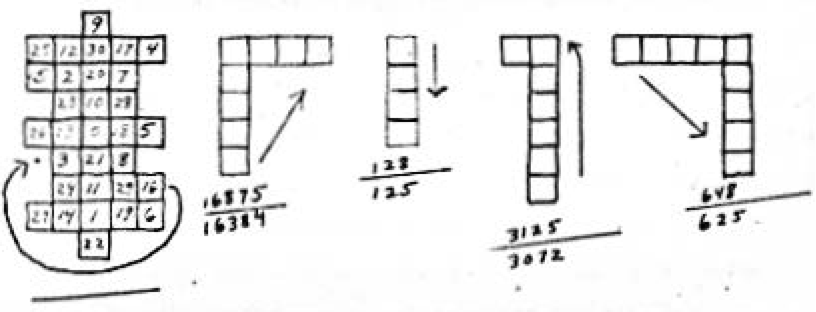
If 12- and 53-tone temperaments are essentially Pythagorean, in that they feature particularly satisfactory fifths, 31-tone temperament is not. The major third is the most satisfactory interval, and of the just intonations offered for 31-tone systems, those that offer the closest approximation to 31-tone equal temperament are the ones which use the greatest number of thirds and the fewest fifths. Among the proponents of 31-tone temperament, at least three, Würschmidt, Ariel, and Fokker, have proposed systems of just intonation involving 31 tones to the octave, as well as methods for defining or deriving the temperament. Würschmidt and Ariel derive the system through thirds and fifths alone, whereas Fokker includes sevenths in a three-dimensional tone-lattice. Würschmidt and Fokker both define the temperament by the intervals which are made to equal zero. For Würschmidt, using two determinants, there are two such intervals. For Fokker, using three, there are three.

According to Würschmidt, one of the two intervals which are made equal to zero to establish 31-tone temperament is the syntonic comma, 81:80. A third is thereby made equal to four fifths minus two octaves. Since this is also true in 12-tone and other temperaments, it does not by itself define the system. Würschmidt's other identity in 31-tone temperament is between eight thirds, and two octaves and a fifth. The interval which is made thereby to become zero is 393216:390625, about 6 cents in size.

Würschmidt makes two diagrams for 31-tone temperament. The first, symmetrical, requires four different constructing intervals. These are 128:125, the difference between three thirds and an octave (this occurs 18 times in the system); 3125:3072, three thirds minus an octave and a fifth (10 times); 16875:16384, three fifths and four thirds minus three octaves (2 times); and the somewhat larger interval 648:625, representing four fifths minus four thirds and an octave or else four minor thirds minus an octave (1 time).

Würschmidt acknowledges that by upsetting the symmetry through the alteration of a single tone (tone number 16 in Example 30) the fourth and largest constructing interval is eliminated, rendering the system presumably stronger. Example 30 shows Würschmidt's original, symmetrical diagram, with an arrow showing the alternate location of tone 16. The tones are numbered in ascending scalewise order, and are placed by fifths and thirds. The four constructing intervals are shown by the accompanying figures. The alteration shown by the arrow eliminates the single 648:625 from the system and removes one of the 10 appearances of the smallest unit, 3125:3072, replacing these with a nineteenth instance of the diesis, 128:125, and a third instance of 16875:16384.

Example 30: Würschmidt's 31-Tone System

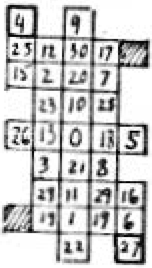


Ariel, in contrast to Würschmidt, makes no use of the interval 16875:16384, but rather uses Würschmidt's three other intervals as follows: 128:125 sixteen times, 3125:3072 twelve times, and 648:625 three times. Revising Würschmidt's diagram the minimum number of times necessary to produce Ariel's system of intervals, it reads as follows in Example 31.

Example 31: Ariel's 31-Tone System

dark border = added by Ariel to Würschmidt's system.

completely dark = deleted by Ariel from Würschmidt's system.



Example 32 shows the properties of the principal 31-tone systems which have been proposed. As the average deviations of the three foregoing systems are within 0.1 of a cent of being identical, only one of them Würschmidt's first system, is included in Example 32. Using Barbour's system of calculating the mean deviation of the constructing intervals, however, the difference between the three methods proposed above is more substantial, Würschmidt's second system involving the smallest mean deviation, 5.2 cents, and Ariel's the greatest, 6.4 cents. These figures compare

Example 32: Chart of Intervals in 31-Tone Just Systems

Unit of System	Equal Temper-ament	Würschmidt	Fokker	Alternate Septimal Tunings	
0	00.00	00.00	00.00		
1	38.71	41.06	27.3	48.7	35.65
2	77.42	70.68	92.17	70.68	84.44
3	116.13	111.74	119.5		
4	154.84	162.85	155.1		
5	193.55	203.91	203.91		
6	232.26	244.97	231.2		
7	270.97	274.58	266.85		
8	309.68	315.64	323.4	301.9	315.64
9	348.39	345.25	359.0		
10	387.10	386.31	386.31		
11	425.81	427.37	435.1		
12	464.52	456.99	470.75		
13	503.23	498.05	498.05		
14	541.94	539.11	(six figures see below)		
15	580.65	568.72	590.22		
16	619.35	609.78*	617.5	Würschmidt:	
17	658.06	660.89	653.2	*or 631.28	
18	696.77	701.95	701.95		
19	735.48	743.01	729.25		
20	774.19	772.63	764.9	786.4	772.63
21	812.90	813.69	821.4		
22	851.61	854.75	857.1		
23	890.32	884.36	884.36		
24	929.03	925.42	933.15		
25	967.74	955.03	968.8		
26	1006.45	996.09	996.09	1017.60	1003.8
27	1045.16	1037.15	1061.0	1039.5	1052.6
28	1083.87	1088.26	1088.26		
29	1122.58	1129.32	1115.6		
30	1161.29	1158.94	1172.7	all others the same	
31	1200.00	1200.00	1200.0	as Fokker's	
Average deviation:		5.18	6.75	6.05	5.17 cents
Average number of steps		2.9	2.55	2.55	2.4
Mean deviation scalewise		5.2	10.4	9.7	7.9

The six different pitches for Fokker's 14th tone are:

525.25
533.7
541.4
545.85
554.6
and 563.95 cents. For calculating the average deviation, the best, 541.4 is used for all three septimal systems shown.

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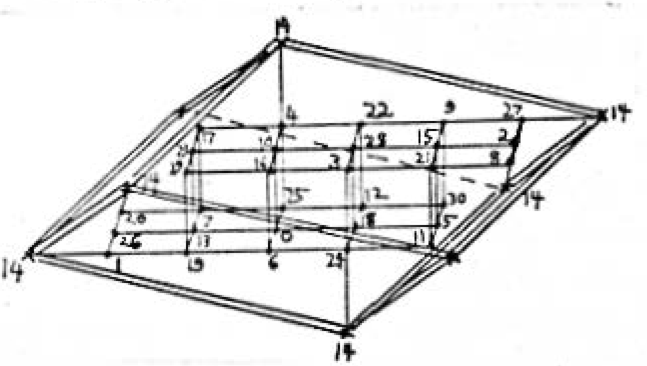
favorably with the mean deviations for just intonation and meantone temperament in a 12-tone system, even allowing for the difference in size of the basic intervals of the two systems.

The diagram of Fokker's tone-lattice is shown below in Example 33.

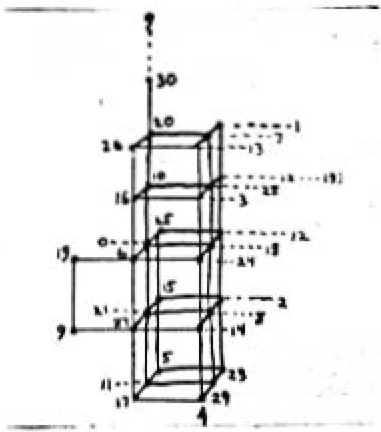
Example 33: Fokker's 31-Tone Tone-Lattice



The numbers Fokker actually used are altered both here and in Example 32 by the deduction of the constant, 2, from every tone.

As in 53-tone temperament, Fokker pays a price for the addition of a third dimension. The intervals of his just system are less accurately portrayed by the equal temperament than are those of Würschmidt's and Ariel's systems. In the Fokker diagram the fifths are horizontal, the thirds vertical, and the sevenths diagonal in order to present an illusion of depth. Why, one might ask, does Fokker use the fifth far more than either of the other intervals when it is the fifth that is the weakest of the three basic intervals in the 31-tone system? The reason does not lie in a Pythagorean aesthetic, since Fokker never, in his theoretical writings, singles out the fifth for special emphasis among the basic intervals. The explanation appears, rather, to lie within the dynamics of the interrelationships of the three basic intervals in the 31-tone system. The diagonal is limited to two spaces in the diagram, because three sevenths would interfere with the formation of fifths and can therefore never be used. Since the seventh is itself created by two thirds plus two fifths, one of these latter intervals must be used only once. Since the third is the better of the two intervals in 31-tone temperament, one might expect Fokker to use only one fifth. Indeed, it is almost possible to reverse the fifths and thirds in Fokker's diagram and still have a complete system. But again Fokker would have been confronted with a limit caused by another identity in the 31-tone system: three thirds plus two sevenths forms a fifth. A system based on one fifth, two sevenths, and two thirds would produce only 3 x 3 x 2 = 18 tones, not nearly enough to complete the system. The remaining tones can be added by further use of thirds and sevenths, if desired, but only in a most un-compact manner. Such a system is shown, below, as Example 34. Its average and mean deviations are markedly smaller than those of Fokker's system,[41] but the absence of compactness and symmetry are serious counterbalancing liabilities.

Example 34: A Three-Dimensional 31-Tone System Built Primarily on Thirds



In the accompanying system, the largest deviation from tone zero to any other tone, as opposed to equal temperament, is approximately 7 cents to tone nine. An alternate tone 9, through which this discrepancy is greatly reduced, is shown by the dotted line.

Fokker's basic system is built on three fifths (a fourth fifth cannot be used regularly as four fifths form an identity with one third), two sevenths, and one third.

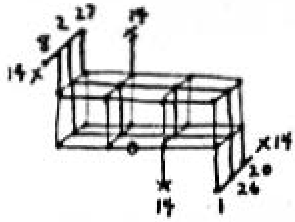
[41] The mean deviation of the system shown in Example 34 is 4.35 cents as compared to 10.4 cents in Fokker's system.

In the diagram, Example 33, this forms a basic parallelopiped containing 4 x 3 x 2 = 24 tones. The two systems listed as "alternate septimal" systems in Example 32, above, use Fokker's basic parallelopiped to account for what might be considered their 24 central tones. The two alternate systems propose alternative methods to Fokker's Pythagorean extension in order to obtain the remaining seven tones (six, not counting the 14th tone which Fokker offers in six different ways) in a manner which will minimize the discrepancies as against equal temperament.

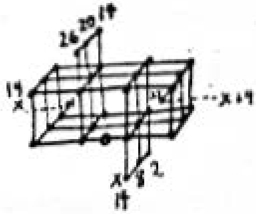
The first alternative system alters each of the six fixed outer tones of Fokker's system by a syntonic comma, resulting in the replacement of Fokker's two sets of three extra fifths (each placed a seventh apart from one another in each set) with two sets of extra thirds (similarly placed with respect to one another). The extra tones in all systems are numbered 1, 2, 8, 20, 26, and 27. The second alternate system draws each of the six outer tones from whatever source it can in order to produce a series of intervals with the lowest possible deviations from equal temperament. The figures at the bottom of Example 32, above, show how great an effect the change of these six tones has on the statistics of deviation. Example 35, below, shows the tone-lattice of each of the two alternate systems which appear in Example 32.

Example 35: Two Alternate Septimal Systems

First Alternate System:



Second Alternate System:



In the systems with many major thirds, the diesis, 128:125, forms the backbone of the scale structure.[42] This interval is completely absent from Fokker's system and the systems shown in Example 35. Because this diesis differs in size from the unit 1/31 by only three cents, those systems which use it show up particularly well with respect to mean deviation which is calculated on the basis of the constructing intervals.

The mean deviation of the single units of the system is a questionable basis for measuring the appropriateness of a multiple-division system such as 31-tone temperament;

[42] Würschmidt uses this diesis 18 times in his symmetrical system and 19 times in his improved system. The system in Example 34 uses this diesis 11 times.

it is highly unlikely that the single unit will be used harmonically, or even with much frequence melodically. By the method of averaging the deviation from equal temperament of all of the intervals of the system from one central or generating tone, a set of figures more representative of the harmonic and melodic accuracy of the tempered system emerges. By this method of calculation the difference in quality between Fokker's and Würschmidt's systems becomes much less, and with the second of the suggested modifications, Fokker's system attains an average deviation as small as Würschmidt's.

Despite the close approximation to 31-tone temperament by the systems based on large numbers of superimposed thirds, Fokker's system, especially with the second of the modifications here proposed, seems superior to any other just 31-tone system. The additional fifths render a service proportionate to the intonational problems they create. The fifth, as the most perfect consonance after the octave, would appear to be an interval required plentifully in any system of multiple division based on an acoustical concept of consonance. The presence of sufficient fifths is shared by Fokker's and Würschmidt's systems, but not by the system shown in Example 34. The sevenths which distinguish Fokker's system from Würschmidt's offer a new dimension and direction for consonance in the 31-tone system at almost no cost in intonational accuracy. With the recommended modifications,[43] Fokker's system would appear to possess sufficient resemblance to the tempered system (which he actually uses in practice) to be of thorough serviceability.[44]

CURRENT PRACTICE IN 31-TONE TEMPERAMENT

Since the construction in the middle 1940's of an organ in 31-tone equal temperament at Teylers Stichting in Haarlem under Professor Fokker's leadership, a growing literature of tricesimoprimal music has developed. Beginning as a strictly local concern, music for the 31-tone organ has increasingly attracted composers from other European countries.[45] Other instruments have been designed recently for 31-tone temperament. Among them is a 31-tone

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[43]The modifications might be objected to on the grounds that Fokker's original system meets at 6 of its ends while the modified systems meet only at 4. One's viewpoint depends, of course, on which features of a tone-system he deems important. To me the modification is thoroughly justified for the more equal intervals which result.

[44]One might ask why, since all the above roads lead to equal temperament, it should make any difference how the system is derived. Practical questions involving intonation are involved, since it can be assumed that singers and players of variable-pitch instruments such as violin and trombone would attempt to come as close as possible to the just intonation. Also, in judging the tempered system it is necessary to know for what purposes it is supposed to be suitable. Finally, it is reasonable to assume that chord- and scale-building within a tempered system will depend on the composer's views, conscious or unconscious, as to the theoretical basis of the system.

[45]Letter from Prof. Fokker, March 9, 1960. The English composer Alan Ridout is cited as an example.

trumpet, an adaptation by Martin Vogel, the author of the definitive work on the seventh partial. Prof. Fokker, in reporting on this instrument in his letter does not describe its mechanism. Presumably a complete re-ordering of valves is required, possibly involving 1, 2, 4, and 8 units of the system respectively.

The first compositions for the 31-tone organ reflect Fokker's preoccupation with Euler's genera even when they are not by Fokker. A competition was held in 1947 for which it was specified that the composers use a set of these genera. The winning work, Mart. J. Lursen's Modi Antichi Musiche Nuove, has been published. Lursen's work reflects the conservative approach to multiple division of the Fokker school. In each of the pieces, only from 4 to 8 of the available pitches are used per octave. The ten genera employed represent all of the threefold combinations of the 3rd, 5th, and 7th partials: 3^3, $3^2 \times 5$, 3×5^2, 5^3, $3^2 \times 7$, $3 \times 5 \times 7$, $5^2 \times 7$, 3×7^2, 5×7^2, and 7^3. Modi Antichi Musiche Nuove consists of 20 very short pieces, among which are included small dances such as a gavotte and a sarabande, several pieces titled after the technical means employed, such as "Bimodal Perpetual Inverted Canon," and a few miscellaneously titled pieces such as "Prelude," "Rondino" and "Inventionetta." Reaching out beyond the limited timbres offered by the 31-tone organ, Lursen specifies varied instrumental groups as ideal for each of the small pieces, should they ever develop the capacity to play them.

A recording of tricesimoprimal music has been released commercially in Europe (Phillips 400090, 45 r/p/m), while additional recordings exist in University libraries.[46] It is hoped that some of these recordings will become more available in this country.

With generally favorable acoustical credentials and with a distinguished precursor in meantone temperament, 31-tone temperament offers one of the most promising realms within the field of multiple division. Under Professor Fokker's nurturing the beginnings of a literature of music have emerged. While it could hardly be considered likely, it is within the scope of the possible that a sudden great increase in the use of this temperament might occur in the near future.

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[46] A recording of tricesimoprimal works by Jan van Dijk, Henk Badings, and Arie de Klein is in the library of the University of Basel, Switzerland. ---Letter from Prof. Fokker.