A GENERALIZATION OF THE CONNECTION BETWEEN THE FIBONACCI SEQUENCE AND PASCAL'S TRIANGLE

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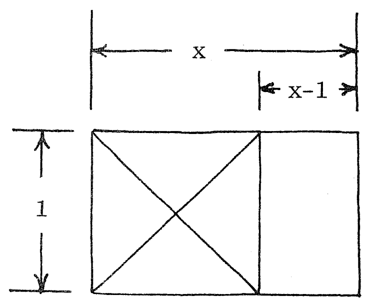
Before the main point of this paper can be developed, it is necessary to review some elementary facts about the Fibonacci Sequence and Pascal's triangle.

It is well-known that rectangles exist such that if a full-width square is cut from one end, the remaining part has the same proportions as the original rectangle.



Assuming width to be unity and length x, we have

$$\frac{1}{x} = \frac{x-1}{1}$$

or

(1) $$x^2 - x - 1 = 0$$

The greatest root of (1) is the number φ, called the Golden Ratio, and the rectangle defined is the Golden Rectangle of Greek geometry. Each root of (1) has the property that its reciprocal is itself diminished by 1, so that

$$\frac{1}{\varphi} = \varphi - 1 \quad .$$

Given any two initial integral terms u_1 and u_2 not both zero, a Fibonacci Sequence is defined recursively by

(2) $$u_n = u_{n-1} + u_{n-2} .$$

It is a well-known property of such sequences that

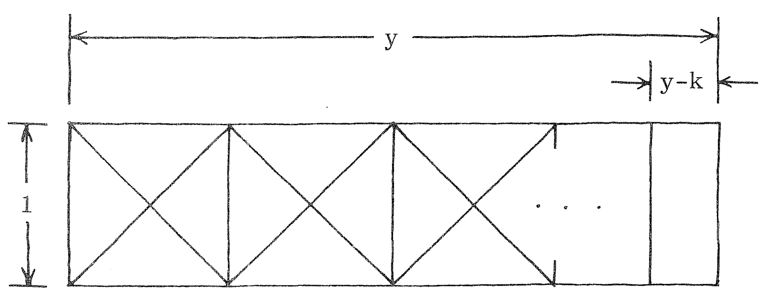
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \varphi .$$

If $u_1 = 0$ and $u_2 = 1$, we have the Fibonacci sequence.

If a rectangle is defined such that when an integral number k of full-width squares are cut from one end, the remaining part has the same proportions as the original rectangle, then

(3) $$y^2 - ky - 1 = 0$$

where the width is unity and the length is y.



The rectangle defined is a golden-type rectangle. The roots of (3) behave much like φ, that is, $1/y = y - k$. The greatest root in absolute value of (3) is the

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} ,$$

where $u_n = ku_{n-1} + u_{n-2}$. In fact, it is well-known that under certain conditions Fibonacci-like sequences defined by

$$u_n = au_{n-1} + bu_{n-2} \tag{4}$$

given initial terms u_1 and u_2 not both zero, where a and b are real, have the property that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha \quad ,$$

where α is the greatest root in absolute value of (See [3])

$$x^2 - ax - b = 0 \tag{5}$$

The condition is that a and b must be such that the roots of (5) are not both distinct, and equal in absolute value.

The above general result can be established in the following way: Consider sequences such that the n^{th} term u_n satisfies

$$u_n = c\alpha^n + d\beta^n \quad . \tag{6}$$

By substitution in (4), α and β can be determined so that sequences (6) will satisfy (4) and be Fibonacci-like sequences. We find that the coefficients of c and d are $\alpha^{n-2}(\alpha^2 - a\alpha - b)$ and $\beta^{n-2}(\beta^2 - a\beta - b)$, respectively. Sequences (6), therefore, satisfy (4) if α and β are roots of (5).

On the other hand, if α and β are roots of (5), then $c\alpha^{n-2}(\alpha^2 - a\alpha - b) + d\beta^{n-2}(\beta^2 - a\beta - b) = 0$ is satisfied for any choice of c and d. But then we have $c\alpha^n + d\beta^n = a(c\alpha^{n-1} + d\beta^{n-1}) + b(c\alpha^{n-2} + d\beta^{n-2})$. Moreover, if $\alpha \neq \beta$, c and d can be determined given initial terms u_1 and u_2. Hence a sequence satisfying (4) satisfies (6) under the conditions stated. If $|\alpha| > |\beta|$, we can use (6) to obtain the

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{c\alpha + d(\beta/\alpha)^n\beta}{c + d(\beta/\alpha)^n} = \alpha \quad .$$

The above limit does not exist, of course, if $\alpha = -\beta$. If the roots of (5) are equal, then we can set

$$u_n = c\alpha^n + nd\alpha^n \tag{7}$$

and show by arguments similar to those above that (7) is a Fibonacci sequence if and only if α is the root of (5) and $a\alpha + 2\beta = 0$. But the roots of (5) are equal if and only if $\alpha = a/2$ and $b = -a^2/4$. Therefore all requirements for (7) being a Fibonacci sequence are met. It is now possible to solve for c and d, and to show that for sequences (7),

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha \quad .$$

An interesting observation has been made about the array of numerals known as Pascal's Triangle. If a particular set of parallel diagonals is designated as in Fig. 1, then the sequence resulting from the individual summations of the terms of each diagonal is the Fibonacci sequence. [2]

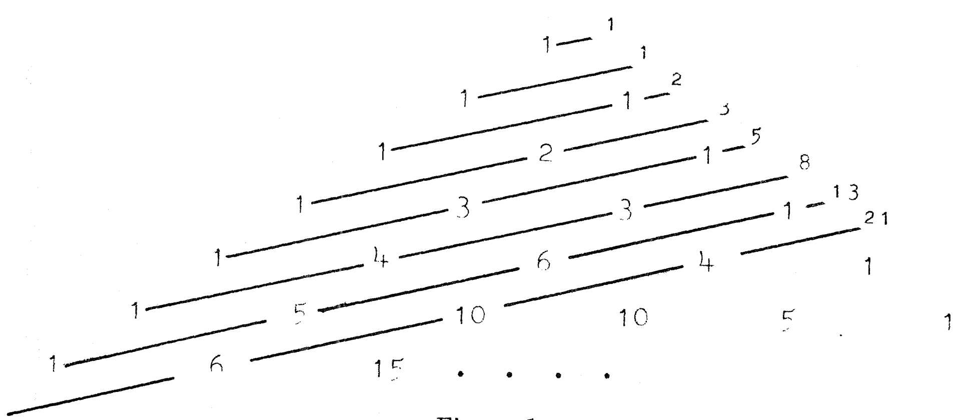


Figure 1

Therefore, the limit of quotients of sums of terms on these parallel diagonals of the triangle is α. We shall now show that some generalizations of this connection can be made.

To begin, we note that the indicated diagonal sums in Fig. 2 are indeed the first few terms (except the first) of (4) if $u_1 = 0$ and $u_2 = 1$.

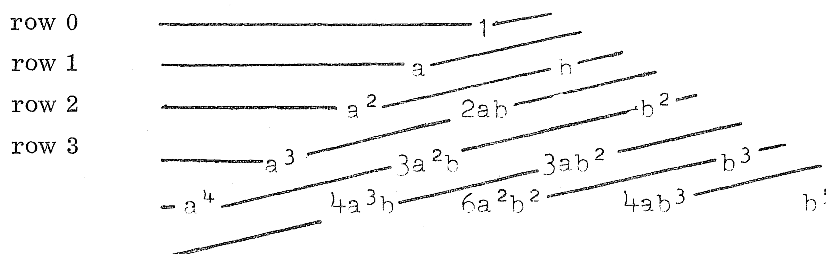


Figure 2

Other sets of parallel diagonals of Fig. 2 also have interesting properties. It is possible to formalize the definition of the array given as Fig. 2, but it will be more efficacious here to simply refer informally to the figure in the arguments to follow. We will assume only that a and b are real, and that Fig. 2 is a Generalized Pascal's Triangle. The row index shall be j, and the term index for each row, δ, each ranging over the non-negative integers. The j^{th} power of $(a + b)$ is the sum of terms in the j^{th} row of Fig. 2.

Definition 1. A diagonal sum x_{jr} of the generalized Pascal's triangle shall be given by

$$x_{jr} = \sum_{\delta=0}^{\left[\frac{j}{r+1}\right]} \binom{j - r\delta}{\delta} a^{j-\delta(r+1)} b^{\delta}$$

Counting from left to right in Fig. 2, the $(\delta + 1)$th term of the diagonal sum is the $(\delta + 1)$th term in the $(j - r\delta)$th row of the triangle as δ ranges over the non-negative integers. Hence x_{jr} is a function of j and r.

Note that the role of r is simply to determine which terms of the triangle are to be summed. This has the effect of defining a set of parallel diagonals for each r. For example, if $r = 1$, the first term of x_{61} is the first term of the sixth row of Fig. 2. The second term of x_{61} is the second term of the fifth row of Fig. 2, and so on. If $r = 3$, the first term of x_{63} is the first term of the sixth row of Fig. 2, but the second term of x_{63} is the second term of the third row, and so on. When $r = 0$, x_{j0} is the sum of terms on the j^{th} row. A sequence $\{x_{jr}\}_j$ of diagonal sums is uniquely determined by r. Since for $j = 0$ the $(j - r\delta)^{th}$ row is defined for every r only when $\delta = 0$, $x_{0r} = 1$ for all r. Further, $x_{1r} = a$ if $r > 0$. If $r = 2$, the first few terms of the resulting sequence are:

$$(1,\ a,\ a^2,\ a^3 + b,\ a^4 + 2ab,\ a^5 + 3a^2b,\ \cdots)\quad .$$

Theorem 1. For sequences $\{x_{jr}\}_j$ of sums of terms on parallel diagonals of the generalized Pascal's triangle,

$$x_{jr} = ax_{(j-1)r} + bx_{(j-r-1)r} \quad . \tag{8}$$

Proof:

$$bx_{(j-r-1)r} + ax_{(j-1)r} = \sum_{\delta=0}^{\left[\frac{j-r-1}{r+1}\right]} \binom{j-r(\delta+1)-1}{\delta} a^{j-\delta(r+1)-(r+1)}b^{\delta+1} + \sum_{\delta=0}^{\left[\frac{j-1}{r+1}\right]} \binom{j-r\delta-1}{\delta} a^{j-\delta(r+1)}b^{\delta}$$

$$= \sum_{\delta=1}^{\left[\frac{j}{r+1}\right]} \binom{j-r\delta-1}{\delta-1} a^{j-\delta(r+1)}b^{\delta} + \sum_{\delta=0}^{\left[\frac{j-1}{r+1}\right]} \binom{j-r\delta-1}{\delta} a^{j-\delta(r+1)}b^{\delta}$$

$$= \sum_{\delta=1}^{\left[\frac{j}{r+1}\right]} \binom{j-r\delta-1}{\delta-1} a^{j-\delta(r+1)}b^{\delta} + a^j + \sum_{\delta=1}^{\left[\frac{j-1}{r+1}\right]} \binom{j-r\delta-1}{\delta} a^{j-\delta(r+1)}b^{\delta}$$

$$= a^j + \sum_{\delta=1}^{\left[\frac{j}{r+1}\right]} \left\{ \binom{j-r\delta-1}{\delta-1} + \binom{j-r\delta-1}{\delta} \right\} a^{j-\delta(r+1)}b^{\delta}$$

but

$$\binom{j - r\delta - 1}{\delta - 1} = \binom{j - r\delta}{\delta} \cdot \frac{\delta}{j - r\delta}$$

and

$$\binom{j - r\delta}{\delta} \cdot \frac{j - \delta(r + 1)}{j - r\delta} = \binom{j - r\delta - 1}{\delta}$$

so

$$bx_{(j-r-1)r} + ax_{(j-1)r} = a^j + \sum_{\delta=1}^{\left[\frac{j}{r+1}\right]} \left\{ \binom{j - r\delta}{\delta} \cdot \frac{\delta}{j - r\delta} + \binom{j - r\delta}{\delta} \cdot \frac{j - \delta(r + 1)}{j - r\delta} \right\} a^{j-\delta(r+1)} b^{\delta}$$

$$= a^j + \sum_{\delta=1}^{\left[\frac{j}{r+1}\right]} \binom{j - r\delta}{\delta} a^{j-\delta(r+1)} b^{\delta} = x_{jr} \quad .$$

In view of Theorem 1, any property of sequences defined recursively by

(9) $$u_n = au_{n-1} + bu_{n-r-1}$$

will be a property of sequences of sums of terms on diagonals of the generalized Pascal's triangle. Further, these diagonal sequences will all be of the special case $u_1 = 0$, $u_2 = 1$, $u_3 = a, \cdots, u_{r+1} = a^{r-1}$; since $r + 1$ initial terms are required for (9) to generate a sequence. We note that diagonal sum $x_{(n-2)r}$ is u_n of (9) given the above initial terms.

As in the proof of

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \varphi \quad ,$$

given (2), we shall establish the existence of similar limits for the sequences defined by (9). If we set

$$(10) \qquad u_n = e_0\alpha_0^n + e_1\alpha_1^n + e_2\alpha_2^n + \cdots + e_r\alpha_r^n \quad ,$$

then substituting in (9) the coefficients of the e_i are

$$\alpha_i^{n-r-1}(\alpha_i^{r+1} - a\alpha_i^r - b)\ (i = 0, 1, \cdots, r) \quad ,$$

and (9) is satisfied if the α_i are the $r + 1$ roots of

$$(11) \qquad x^{r+1} - ax^r - b = 0 \quad .$$

Conversely, given that the α_i are the roots of (11), it follows that sequences (9) can be written in the form of (10) if the e_i can be determined. One can obtain from the given $(r + 1)$ initial terms $(r + 1)$ equations $u_j = e_0\alpha_0^j + e_1\alpha_1^j + \cdots + e_r\alpha_r^j$ $(j = 1, 2, \cdots, r + 1)$. This set of equations has a non-trivial solution for the e_i, however, if and only if the α_i are distinct. Whether or not the terms of sequences defined by (9) can be written in the form of (10) depends, therefore, on whether or not we can determine conditions for the multiplicity of the roots of (11).

Suppose p is a root of (11) where a and b are both not zero. Then (11) may be written as $(x - p)Q(x) = 0$ where

$$Q(x) = x^r + (p - a)x^{r-1} + (p - a)px^{r-2} + (p - a)p^2x^{r-3} + \cdots + (p - a)p^{r-1} \, .$$

Clearly p is a multiple root of (11) if and only if it is a root of $Q(x) = 0$. But then it is easily verified that

$$p = \frac{ar}{r+1} .$$

Now since p is real, at least all complex roots of (11) are distinct.

DeGua's rule for finding imaginary roots states that when 2m successive terms of an equation are absent, the equation has 2m imaginary roots; and when 2m - 1 successive terms are absent, the equation has 2m - 2 or 2m imaginary roots, according as the two terms between which the deficiency occurs have like or unlike signs. Accordingly, we see that (11) has at most three real roots, since there are r - 1 successive absent terms and hence at least r - 2 complex roots. Further, if $f(x) = x^{r+1} - ax^r - b$, the two critical numbers of f are zero and $ar/(r+1)$. Since $f(ar/(r+1))$ is an extremum of f, the greatest multiplicity of any real root of (11) is two. [1]

If b is zero but a is not, then the roots of (11) are zero (of multiplicity r), and a. Other cases are trivial.

If the real roots of (11) are distinct and α_0 is any root such that $|\alpha_0| > |\alpha_i|$ $(i = 1, 2, \cdots, r)$, then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{e_0\alpha_0^{n+1} + e_1\alpha_1^{n+1} + \cdots + e_r\alpha_r^{n+1}}{e_0\alpha_0^n + e_1\alpha_1^n + \cdots + e_r\alpha_r^n}$$

$$= \lim_{n\to\infty} \frac{e_0\alpha_0 + e_1\alpha_1(\alpha_1/\alpha_0)^n + \cdots + e_r\alpha_r(\alpha_r/\alpha_0)^n}{e_0 + e_1(\alpha_1/\alpha_0)^n + \cdots + e_r(\alpha_r/\alpha_0)^n} .$$

Therefore

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_0 .$$

It is clear that $ar/(r+1)$ is a root of (11) if and only if

$$b = -\frac{a^{r+1}r^r}{(r+1)^{r+1}} .$$

Suppose α_0 and α_1 are this root. Then we can set

(12) $$u_n = e_0\alpha_0^n + ne_1\alpha_0^n + e_2\alpha_2^n + \cdots + e_r\alpha_r^n$$

and use (9) to find the coefficients of the e_i. The coefficient of e_i where $i \neq 1$ is $\alpha_i^{n-r-1}(\alpha_i^{r+1} - a\alpha_i^r - b)$ and for e_i we have

$$n\alpha_0^{n-r-1}\left(\alpha_0^{r+1} - a\alpha_0^r - b + \frac{a\alpha_0^r}{n} + \frac{b(r+1)}{n}\right) .$$

It is clear that the required condition is that the α_i be the roots of (11) and $a\alpha_0^r + b(r+1) = 0$. But with α_0 chosen as above, this is indeed the case. As before, (12) can be used to generate equations which enable us to find the e_i. Finally

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

exists and is the greatest root of (11) in absolute value.

Since (9) generates a real sequence given real initial terms, not only is

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

the greatest root of (11) in absolute value, but it must also be real. Hence the greatest root in absolute value of (11) must be real.

If a, b, and r in (11) are such that two distinct roots share the greatest absolute value of all roots, then it is easily shown that no limit exists.

Employing simple unit theorems, we can prove that

$$\lim_{n\to\infty} \frac{u_{n+s}}{u_n} = \alpha_0^s \text{ if } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_0 .$$

We are now able to state that:

Theorem 2. For all sequences formed by sums of terms on parallel diagonals of the generalized Pascal's triangle, and for all sequences defined by (9) given $r + 1$ initial terms,

$$\lim_{n \to \infty} \frac{u_{n+s}}{u_n}$$

exists and is the greatest root in absolute value of

$$x^{\frac{r+1}{s}} - ax^{\frac{r}{s}} - b = 0 \quad ,$$

provided this absolute value is not shared by two distinct roots.

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