27 Meadowbrook Rd. Apt. 3
Toronto 399, Ontario
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To whom it may concern:

I send this to you mainly because you seem the highly appropriate place to send it, and in the hope that it may fall perhaps (or be directed) into some actively interested hands. I should very much like to be in on any future development.

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Golden Tones?

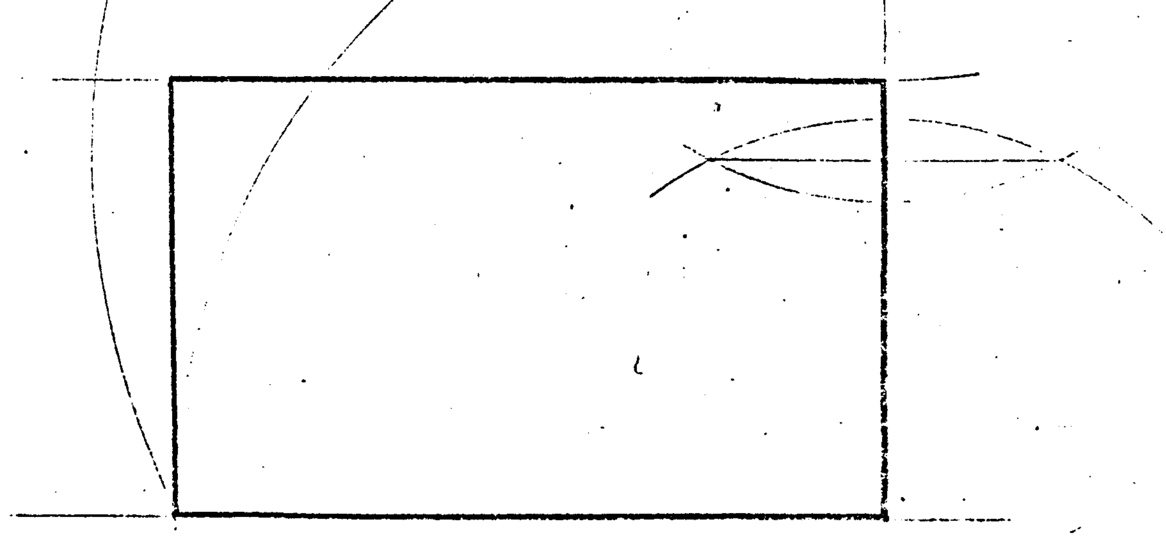
a) APOLOGY AND INTRODUCTION

Not much of what I am about to present has any great semblance of depth; it could have been proposed by just about any interested doodler, and in fact was discovered by me in just that way when I was in grade eleven. My preoccupation then was with the unusual arithmetical and mystical (or mythical) artistic properties of the "golden ratio", and what seems now to be the major justification for what follows (the idea of "maximal dissonance") did not follow for some time to come. Even so, this letter is still long delayed. I write it now (obtuse though it may yet be) to express my eagerness to discuss more fully any of the ideas presented with anyone interested enough to want to bother.

b) THE INTERVAL

It is easy to observe that two tones are consonant exactly to the degree that their harmonics overlap. For example, the unison is the most consonant interval because all the harmonics of both tones coincide; in the octave only the odd harmonics of the lower tone are unpaired; in a perfect fifth the even harmonics of the second tone match every third one of the first; and so on, the intervals becoming less and less consonant. There are, in fact, though, intervals which sound relatively consonant but for which none of the harmonics involved precisely coincide. These are the ones for which the overtones come close to matching, and this is precisely what happens, for example, in even tempering. For this reason, the question of whether there is a "most" dissonant interval is a rather subtle one. What we desire is not only a ratio which results in few or no coincident harmonics, but much more, the harmonics should "overlap" as little as possible, and least for lower ones.

Very auspiciously, there is a <u>unique</u> ratio for which the worst possible things happen; it was well known to the ancient Greeks, though not at all by this characterization; it was intensively studied by Fra Lucia Pacioli about the end of the 15th century, still probably unaware of this property; it is called the golden ratio, ~~and divine proportion~~, divine proportion, or extreme-and-mean ratio; it appears in many geometrical, arithmetical and perhaps even some mathematical contexts; it was used extensively by Greek and medieval artists and architects because of its fabled (with a fair dash of Pythagoraean mysticism) property of being the most artistically pleasing proportion of all. (for example for the sides of a rectangle:



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It also appears in natural contexts in plants, for example in the whorls of a sunflower; for example in the arrangement of the leaves of many plants about their stem, the angle between two successive leaves being almost exactly this fraction of a whole circle, precisely, according to one theory, 'because' this is *exactly* the uniform arrangement required so that the leaves *overlap one another the least* and shade each other least from the sun.

We introduce this number in its two guises, first as the golden section or extreme-and-mean ratio:

> "that ratio in which a line segment is divided such that the ratio of the smaller part to the larger part is the same as that of the larger part to the whole",

and second as the number whose continued-fraction expansion is slowest to converge:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}}}}}$$

This number we shall denote by $\frac{1}{\tau} = \frac{1}{2}(\sqrt{5}-1) = .618033989\ldots$; since it is my preference to express things in terms of $\tau = \frac{1}{2}(\sqrt{5}+1) = 1+\frac{1}{\tau} = 1.618033989\ldots$ , as well as this being the convention.

c) A SCALE

Hopefully, it by now seems quite reasonable to suggest that it would be of considerable interest to investigate tonal sequences whose fundamental period is the interval $1:\phi$, by analogy and opposition with the diatonic and twelve-tone scales, whose fundamental period is the most consonant of proper intervals, the octave (the ratio 1:2). Corresponding to the sequence of tonics separated by octaves, we have the sequence $1, \phi, \phi^2, \phi^3, \phi^4, \ldots$, which has the (perhaps interesting) property that each successive term is the sum of the two preceeding it.

We now seek, therefore, to interpose some degrees of a scale in the fundamental interval $1:\phi$. As the Pythagoraeans discovered, however, it is highly likely that the choice is somewhat arbitrary. Still, we shall be guided in our search by the considerations which led (in the eleventh grade) to the investigation of this whole mess in the first place.

Sequences of numbers involving $\phi$ are notorious for their rich combinatorial properties and the striking relationships which 'unexpectedly' appear among the terms. Now, sums and differences ought to have an implicit role in any phenomenon involving the way frequencies combine, because the process involved there is precisely that of heterodyning: feed any two frequencies into a non-linear device (like the ear, for instance) and out comes, along with the original two, both their <u>sum</u> and their <u>difference</u>! This alone suggested originally (together with the artistic mystique of $\phi$) that sequences involving $\phi$ should have some properties of musical interest.

From this point it took a few months of marginal doodling to achieve anything near satisfactory. The main idea was to posit a number satisfying an 'interesting' simple relationship when its $\phi$-multiples were introduced into the sequence of tonics; and to hope that one could choose a number of these relationships to get a scale which was nicely distributed, its degrees neither too plentiful nor too crowded in places (say closer than a semitone) nor too sparse in others. Because a number is considered more suitable precisely the more ways it can be characterized, because the original rationalizations were somewhat vague and are almost lost in antiquity anyway, because the reasons are not really very good and probably quite boring, I choose not to actually display the defining relationships (There are three, sort of.). Suffice us here to display the resultant pentatonic sequence:

$$1\ , \quad \frac{\phi^3}{\phi^2+1}\ , \quad \frac{2}{\phi}\ , \quad \frac{\phi^2}{2}\ , \quad \frac{\phi^2+1}{\phi^2}\ , \quad \phi\ .$$

1.000    1.169...    1.236...    1.309...    1.382...    1.618...

    1.169...    1.056...    1.059...    1.056...    1.169...    (successive intervals)

(Note the reappearance of the octave!) Because of the symmetry, there are only three different intervals involved between successive degrees; with an even-tempered semitone having ratio 1:1.0594 and whole tone ratio :1.122462, two of these intervals are just about a semitone and the other is less than a minor third (untempered: 5:6 = 1:1.2).

d) AFTERTHOUGHTS

i) The degrees of this scale were chosen for their properties in combinations with the tonics rather than for interactions among themselves. This tends to suggest that the scale will be "tonic-centred".

ii) There is a geometrical hint that $\frac{\mathrm{I}^3}{\mathrm{I}^2+1}$ might play a role somewhat akin to that of a dominant.

iii) The claim is that it is precisely the interval of ratio $1:\mathrm{I}$ which is "maximally" dissonant. As well, (or moreover) the harmonics involved here interweave in an especially peculiarly <u>uniform</u> manner. This suggests that the dissonance involved, as part of being maximal, will be <u>uniquely</u> "white".

iv) Observe that none of the conjectures made has ever been tested, nor does anyone know for certain what either the fundamental interval or the proposed scale sounds like.

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