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MANDELBAUM, Mayer Joel, 1932–
MULTIPLE DIVISION OF THE OCTAVE AND THE TONAL RESOURCES OF 19-TONE TEMPERAMENT.

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Music

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principle: better but still not entirely satisfactory.[11] Finally he works out a formula whereby the two processes are alternated, the choice between them to be determined by which of the two would involve the larger devisor. The result of this complicated procedure is almost identical with Brun's somewhat similar calculation based on ternary continued fractions but using subtraction instead of Barbour's division. Barbour's method is available to all who are interested; as Brun's is generally unknown in this country, it will be examined in considerable detail.

PROFESSOR BRUN'S METHOD

The most impressive mathematical method yet proposed for the derivation of systems of equal temperament from a series of just intervals (or from any other intervals if they should be desired) was first presented in 1919 by Viggo Brun, a Norwegian mathematics professor. He has since expanded on his views in a series of articles, the most recent and complete of which will appear this year in the Nordisk Matematisk Tidskrift. I am convinced that it provides the music theorist with the most useful apparatus

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[11]The Jacobi expansion tends to favor temperaments with excellent thirds but questionable fifths: 3, 25, 28, 31, 87. The last two terms listed have good fifths as well. The expansion through the reversal of terms favors the fifths: 1, 2, 5, 12, 41, 53... The mixed expansion alternates on a principle similar to Brun's where the largest of the possible terms is used for each new stage of the expansion.

yet devised for probing the reaches of multiple systems of equal temperament.

Brun's method employs Euclidean algorithms. To begin with, the logarithms of all just intervals on which the system is to be based are computed. Any number of intervals may be employed (although Brun concedes that the method begins to lose its accuracy if too many factors are involved) and any members of the harmonic series may, for experimental purposes be omitted. Let us follow Brun's process, using the octave, fifth, and major third as values for our example.

He calls the logarithms of the three values a, b, and c respectively, and places them, <u>in order from the highest to the lowest</u>, in the left hand column. The right hand column shows values for x_1, y_1, z_1, x_2, y_2, z_2, and x_3, y_3, and z_3, as shown in Figure 1. These represent the number of times the ratio whose logarithm is on the left will go exactly into the ratios of the intervals x, y, and z, which in this case represent the octave, fifth, and major third. Figure 2 shows the same formula with numerical values.

FIG. 1

a	x_1	y_1	z_1
b	x_2	y_2	z_2
c	x_3	y_3	z_3

FIG. 2

(2:1)0.3010	1	0	0
(3:2)0.1761	0	1	0
(5:4)0.0969	0	0	1

Brun's basic process is to substitute "a - b" for "a" in the left hand column, producing with each stage logarithms representing smaller intervals, while in the right hand columns he substitutes "x_1 plus x_2" for "x_2," which is to say that he increases the number of times the intervals (now smaller) are used. Throughout the process, a, b, and c represent the sizes of intervals while x_1, x_2, and x_3, etc., represent the number of occurances of each of the respective intervals such that ax_1 plus bx_2 plus cx_3 equals exactly an octave, ay_1 plus by_2 plus cy_3 a perfect fifth, etc.

FIG. 3

a - b	x_1	y_1	z_1
b	x_1+x_2	y_1+y_2	z_1+z_2
c	x_3	y_3	z_3

The substitution shown in Figure 3 represents no basic change in the values, since $(a - b)x_1$ plus $b(x_1 + x_2) =$ $ax_1 - bx_1 + bx_1 + bx_2 =$ $ax_1 + bx_2$, exactly the same as in Figure 1.

FIG. 4

0.1249	1	0	0
0.1761	1	1	0
0.0969	0	0	1

The smallest interval, c, is a kind of carried-over remainder which remains outside of the main calculations until one of the other figures becomes smaller, whereupon it is renamed b, and the new smallest term becomes the remainder. In Figure 4, the numbers in Figure 2 are modified according to the process shown in Figure 3. In place of the octave, log (a - b) representing the perfect fourth, appears. The x column to the right of the line shows that one fourth plus one fifth comprises an octave. In Figure 5, the terms of Figure 4 are simply rearranged to place the three intervals in positions a, b, and c according to size. In Figure 6, the basic process is repeated. The fourth, b, is deducted from the fifth, a, leaving "a - b," a 9:8 major second. The octave is shown to consist of two fourths and a major second, the fifth of one fourth and a major second. The third is yet to play a role in the process.

FIG. 5

(3:2)0.1761	1	1	0
(4:3)0.1249	1	0	0
(5:4)0.0969	0	0	1

FIG. 6

0.0512	1	1	0
0.1249	2	1	0
0.0969	0	0	1

In Figure 7, the terms in Figure 6 are rearranged to place the interval sizes in descending order. In this process, the major third is advanced to position b. Figure 8 shows a repetition of the basic procedure, with the diatonic semitone 16:15 resulting from the subtraction of 5:4 from 4:3. As this is the smallest interval so far produced, it assumes position c in the rearrangement which is depicted by Figure 9.

FIG. 7

(4:3)0.1249	2	1	0
(5:4)0.0969	0	0	1
(9:8)0.0512	1	1	0

FIG. 8

0.0280	2	1	0
0.0969	2	1	1
0.0512	1	1	0

In Figure 9, the octave is shown as comprised of two just major thirds, a major tone, and two diatonic semitones. This can be stated as follows: $\frac{5}{4}x\frac{5}{4}x\frac{9}{8}x\frac{16}{15}x\frac{16}{15} = \frac{2}{1}$. In Figure 10, it is further broken down into traditional units of the just diatonic scale, three 9:8 tones, two 10:9 tones, and two 16:15 semitones. The other right hand columns show the traditional breakdowns of third and fifth among these intervals.

Figure 11 shows the necessary rearrangement to restore the values on the left hand side to numerical order, and Figure 12 shows the next step in the subdivision with a new small value, the syntonic comma, emerging to assume the role of "c" for quite some time.

FIG. 9

(5:4) 0.0969	2	1	1
(9:8) 0.0512	1	1	0
(16:15)0.0280	2	1	0

FIG. 10

(10:9) 0.0457	2	1	1
(9:8) 0.0512	3	2	1
(16:15)0.0280	2	1	0

FIG. 11

0.0512	3	2	1
0.0457	2	1	1
0.0280	2	1	0

FIG. 12

(81:80)0.0055	3	2	1
(10:9) 0.0457	5	3	2
(16:15)0.0280	2	1	0

We are now at the point where recognizable tempered systems are about to emerge, and it is time to ask ourselves how one is to interpret the figures obtained. As has been stressed, $ax_1 + bx_2 + cx_3$ will always equal an octave. Do we, then, add the number of terms in the first column to the right of the line to obtain the number of tones in our temperament? No; this is not done, because the third value is external to the main process and is likely to be so small as to be negligible to the process as a whole.

FIG. 13

(10:9)	0.0457	5	3	2
(16:15)	0.0280	3	1	0
(81:80)	0.0055	3	2	1

In Figure 13 (which is a simple rearrangement of Figure 12) the octave is composed of ten units, three of which are mere syntonic commas. To propose a 10-tone temperament with units representing everything from minor tones (10:9) to syntonic commas (81:80) would be absurd. What Figure 13 does offer, in its top row, (derived from row 2 of Figure 12) is an instance of 5-tone temperament in which the fifth and major third have their proper values. A comparison of this row with the sum of the terms in Figure 9, shows the superiority of this process. The sum of the terms in Figure 9 are 5, 3, and 1 respectively. If this were indeed supposed to represent 5-tone temperament, the third would be much too small. 386 cents is considerably closer to 480 cents than it is to 240 cents; the row 5, 3, 2 representsthis temperament better than the rougher, earlier estimate, 5, 3, 1.

Figure 14 carries Brun's process a step farther, with a new temperament of 7 tones to the octave emerging.

Figure 15 represents the characteristic rearrangement of terms, and Figure 16 carries the process another step, showing the emergence of 12-tone temperament. It might be mentioned at this point that the progression from 5- through 12- has begun to follow Yasser's Fibonacci series. This it will continue to do as long as every new interval value (a - b) remains larger than the syntonic comma, which is presently value c. As soon, however, as the new value has been reached which is smaller than the syntonic comma, the syntonic comma will pass from the role of "defining" interval to that of "constructing" interval and the values to the right of it in the chart will play a role in determining the number of tones in the temperament. It is Figure 15 which best shows the 12-tone system under Brun's approximation. In the octave are seven diatonic semitones (16:15) and five chromatic semitones (25:24). Three syntonic commas are "left over," which are generally added to three of the diatonic semitones in the theory of just intonation.

FIG. 14

0.0177	5	3	2
0.0280	7	4	2
0.0055	3	2	1

FIG. 15

(16:15)0.0280	7	4	2
(25:24)0.0177	5	3	2
(81:80)0.0055	3	2	1

FIG. 16

0.0103	7	4	2
0.0177	12	7	4
0.0055	3	2	1

Figure 17 represents the rearrangement of Figure 16 in the descending order of the intervals, and Figure 18 represents the next step: 19-tone temperament. The syntonic comma remains the smallest figure, but the new interval, representing the difference between the chromatic semitone and the diesis of the major thirds, 128:125, is nearly as small.

Figure 19 shows the rearrangement, and Figure 20 the next step, whereby 31-tone temperament is shown, and where a new interval, smaller than the syntonic comma is formed. It is this stage which alters the Fibonacci series, as 81:80 assumes position b in the rearrangement which is shown in Figure 21.

Thus it is not 19, but rather 3 tones, representing the number of syntonic commas carried by the simpler systems as a remainder, which must be added as the syntonic comma is advanced to the rank of constructing interval. Figure 22 shows the emergence

FIG. 17

.0177	12	7	4
.0103	7	4	2
.0055	3	2	1

FIG. 18

.0074	12	7	4
.0103	19	11	6
.0055	3	2	1

FIG. 19

.0103	19	11	6
.0074	12	7	4
.0055	3	2	1

FIG. 20

.0029	19	11	6
.0074	31	18	10
.0055	3	2	1

FIG. 21

.0074	31	18	10
.0055	3	2	1
.0029	19	11	6

of 34 as the next system through the combining of 31- and 3-tone temperaments, the latter shown, by Brun's method, to have the best features to balance the weaknesses of 31-tone temperament.

Figure 23 shows the rearrangement which follows; a new smallest interval assumes position c. In Figure 24, it is 53-tone temperament which is shown to arise out of the combination of ultrapositive 34- and negative 19-tone temperaments. With the evolving of 53-tone temperament, the syntonic comma has completed its role as constructing interval and will itself be subdivided.

Figure 25 represents a rearrangement and Figure 26 introduces the next temperament of the series, possessing 87 tones to the octave. The excellent merits of this system are considered briefly in Chapter 6, above.

FIG. 22

.0019	31	18	10
.0055	34	20	11
.0029	19	11	6

FIG. 23

.0055	34	20	11
.0029	19	11	6
.0019	31	18	10

FIG. 24

.0026	34	20	11
.0029	53	31	17
.0019	31	18	10

FIG. 25

.0029	53	31	17
.0026	34	20	11
.0019	31	18	10

FIG. 26

.0003	53	31	17
.0026	87	51	28
.0019	31	18	10

The next temperament to emerge, after the rearrangement shown in Figure 27, is 118-tone temperament in Figure 28. This is the "perfect system" of so many advocates of temperaments based on fifths and thirds, and it is also shown in Chapter 6.

FIG. 27

.0026	87	51	28
.0019	31	18	10
.0003	53	31	17

FIG. 28

.0007	87	51	28
.0019	118	69	38
.0003	53	31	17

The algorithm can be continued, in which case it yields 205, 323, 441, and 494-tone temperaments. The temperaments agree almost completely with those derived by Ariel and by Barbour by different methods, with the latter members of the series showing the greatest divergence between them. The error of third and fifth alike in 441-tone temperament is less than .05 cents and compares well with the errors in systems of many tones which are advocated by other theorists.

APPLYING BRUN'S PROCESS TO OTHER INTERVALS

A process similar to the one outlined above can be followed for any combination of intervals. The theorist may choose to use additional values in the harmonic series. Brun himself has carried the process through using the above values plus the seventh partial, obtaining as temperaments for the octave, 13-, 15-, 18-, 31-, 35-, 53-, and 68-tones.[12] The agreement of this series with the one studied above at the 31- and 53- levels strengthens the case for these temperaments. Brun has also used the same process with all partials through 11, and obtained the following results: 15, 17, 22, 37, 41, 63, 72.... Of these, his own preference lies with 41- and 72-tone temperaments, both of which have been found quite attractive by other theorists, usually for very different reasons.[13] Carrying this algorithm farther, I find the next temperaments to be 89-, 161-, 233-, and 270-tone temperaments. The intervals of 270-tone temperament show the potential value of Brun's method for locating exceptional temperaments.

Interval	3:2	5:4	7:4	11:4
Size of interval	702.22	386.67	968.89	551.11
Size of error	0.27	0.36	0.1	0.2

The smallest unit of the system being 4.44 cents, the error never exceeds 18 percent of the possible; this is a very good achievement for a system with so many disparate factors.

It is also possible to use Brun's method with

[12] In my own calculations using Brun's methods, I arrive at slightly different figures: 49 instead of 35, and 71 instead of 68.

[13] 41-tone temperament is discussed above in the first section of Chapter Five, and 72-tone temperament is discussed near the end of Chapter Four.

intervals lying outside of the harmonic series, should it be the conviction of anyone seeking a more complex equal temperament that substitute intervals (such as a larger major third) are preferable. The method can also be used with members of the harmonic series but omitting one or more, as Yasser suggests with regard to the 3rd partial. Brun's procedure, in short, provides a means for converting any set of preferred intervals into a series of equal temperaments tending to increase in accuracy as more tones are added. There is a shortcoming, in that there is no room for differentiating the importance of the various intervals involved. This same weakness has been noted in other methods for calculating advantageous temperaments. A further shortcoming, if one wishes to call it that, is that it still does not perform a musician's thinking for him. He must still decide which intervals he is to use, and he must select from the series which the method he employs yields that single system which is the best compromise, for his purposes, between accuracy and expediency.